A1 Solving Equations

Solving an equation in one variable, say $x$, can be converted into finding a root of a function $f$, if we rewrite the equation to have 0 on the right-hand side.

A1.1 Bisection Method

Probably the simplest idea to finding the root to a function is the bisection method. Suppose you want the root of $f$. Start with an interval $[a, b]$ such that $f(a)$ and $f(b)$ have opposite signs. If $f$ is continuous on the interval, then by the Intermediate Value Theorem, $f$ is guaranteed to have a root between $a$ and $b$. The bisection method calculates the midpoint of the interval, and retains that half of the interval where a root is guaranteed to be.

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A1.2 The Secant Method

An iterative method for solving equations is one that calculates a sequence of points $x_0, x_1, x_2, \ldots$, each an approximation to the root.

Consider starting with two points, say $x_0$ and $x_1$. A natural idea is to always look at the line segment joining $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$, called a secant. Where the secant crosses the $x$-axis is a new approximation $x_{i+1}$ to the root, hopefully better. This yields the secant method.

There is a related idea, called regula falsi, which might be thought of as a hybrid between the bisection method and the secant method. In regula falsi, a new point is
generated when the secant crosses the axis. However, like in bisection method, the interval guaranteed to contain the root is retained.

A1.3 Errors

If $p^*$ is an approximation to the actual root $p$, then the **absolute error** is $|p - p^*|$ and the **relative error** is

$$\frac{|p - p^*|}{|p|}$$

(provided $p$ is not zero).

Ideally, we would like our process to terminate when the absolute or relative error is within a certain threshold. Of course, since we don’t know the actual answer, we cannot work out the error exactly.

In the bisection method we can certainly bound the absolute error. In the secant method it is common to stop when the absolute difference between two consecutive approximations is below some threshold. However, it is possible to concoct a function where the result is nowhere near the actual root. (Exercise to reader: draw such a situation...)

A1.4 Newton’s Method

In the method called Newton’s Method or **Newton–Raphson**, rather than using a secant from the last values, we use the tangent at the previous value.

Thus the formula is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Of course, that poses the problem of calculating the formula for the derivative.