12.1 3D Coordinates

The three-dimensional coordinate system is the set of points \((x, y, z)\). This geometry has the right-hand rule: if we have usual \(x, y\)-axes on the page, then positive \(z\) is up out of the page. The entire space is sometimes denoted by 
\[ \mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \] 
(where \(\mathbb{R}\) denotes the set of all real numbers).

The distance between two points \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) is
\[ \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2} \]
by Pythagoras.

The projection of a point \(A\) onto a plane is the point on the plane closest to \(A\) (and the line segment joining these is perpendicular to the plane). For example, projecting point \((x, y, z)\) onto the \(xy\)-plane gives \((x, y, 0)\).

Often, one equation gives a surface. (More correctly, the set of all solutions to the equation forms a surface.)

For example, \(x = 3\) is a plane A sphere with center \((h, k, \ell)\) and radius \(r\) is given by
\[ (x - h)^2 + (y - k)^2 + (z - \ell)^2 = r^2 \]
An example of a cylinder is \(x^2 + y^2 = 1\) (where there is no restriction on \(z\)).
12.2 Vectors

A vector has both magnitude and direction. We write $\mathbf{u}$ or $\vec{u}$ or $\langle x, y, z \rangle$.

The vector $\langle x, y, z \rangle$ is called the position vector of the point $(x, y, z)$. A scalar is only a single value (for us a real number).

Vector arithmetic. Vector addition is the net effect, that is, $\vec{a} + \vec{b}$ represents $\vec{a}$ followed by $\vec{b}$. The parallelogram law says that $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.

In algebra:

\[
\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle
\]
\[
\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle
\]
\[
c \langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle
\]

Important example: A vector of magnitude $r$ with angle $\theta$ to the $x$-axis is given by $\langle r \cos \theta, r \sin \theta \rangle$.

Special vectors: $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$ and $\mathbf{k} = \langle 0, 0, 1 \rangle$. So $\langle a_1, a_2, a_3 \rangle = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$. The vector with magnitude $0$ is denoted $\mathbf{0}$.

The length or magnitude of a vector $\langle a_1, a_2, a_3 \rangle$ is $\sqrt{a_1^2 + a_2^2 + a_3^2}$. A unit vector has unit length. To obtain a unit vector with the same direction as $\mathbf{u}$, scale by the inverse of the magnitude:

\[
\frac{\mathbf{u}}{|\mathbf{u}|}
\]

The net effect of forces (called the resultant force) is their vector sum.
12.3 Dot Product

The dot product of two vectors is a scalar and is given by

\[ \langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 \]

The dot product obeys some rules such as being commutative, and it distributes over addition. Note that \( \mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 \).

The big result:

\[ \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta \]

where \( \theta \) is the angle between them. Two vectors are perpendicular/orthogonal if and only if their dot product is 0.

For example, calculate the angle that the major diagonal of a cube makes with a face. Vector for diagonal is \( \mathbf{u} = \langle 1, 1, 1 \rangle \) and vector for diagonal of face is \( \mathbf{v} = \langle 1, 1, 0 \rangle \). So \( \mathbf{u} \cdot \mathbf{v} = 2 \). But \( \mathbf{u} \cdot \mathbf{v} = \sqrt{2} \sqrt{3} \cos \theta \). So \( \theta = \cos^{-1}(\sqrt{2/3}) \).

The scalar projection of vector \( \mathbf{b} \) onto \( \mathbf{a} \) (also known as the component of \( \mathbf{b} \) along \( \mathbf{a} \)) is

\[ \text{comp}_a \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} \]

The vector projection is obtained by multiplying the scalar projection by the unit vector in direction of \( \mathbf{a} \).

As an application, it is known that work is the dot product of force and displacement.
12.4 Cross Product

The cross product of two vectors is a vector and is defined as

\[
\langle a_1, a_2, a_3 \rangle \times \langle b_1, b_2, b_3 \rangle = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle
\]

The cross product is defined so that \( \mathbf{a} \times \mathbf{b} \) is orthogonal to the plane defined by \( \mathbf{a} \) and \( \mathbf{b} \). It can be thought of as a determinant:

\[
\mathbf{a} \times \mathbf{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}
\]

Facts include:

\[
\mathbf{a} \times \mathbf{a} = 0 \quad \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \quad \mathbf{i} \times \mathbf{j} = \mathbf{k}, \ \mathbf{j} \times \mathbf{k} = \mathbf{i}, \ \mathbf{k} \times \mathbf{i} = \mathbf{j}
\]

and distributive laws.

Note that

\[
\mathbf{a} \times \mathbf{b} = 0 \text{ if and only if } \mathbf{a} \text{ and } \mathbf{b} \text{ are parallel.}
\]

And that

\[
|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \theta
\]

which is the area of the parallelogram defined by \( \mathbf{a} \) and \( \mathbf{b} \).

The triple product \( |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| \) gives the volume of the parallelepiped defined by \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \). (Note that \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \).)
12.5 Lines and Planes

**Lines.** There are several formulas/equations for lines. A line can be written as a *parametric equation* in two ways: Either

\[ \mathbf{r} = \mathbf{r}_0 + t \mathbf{v} \quad \text{with } t \in \mathbb{R} \]

for some fixed point \( \mathbf{r}_0 \), fixed vector \( \mathbf{v} \) or

\[ \langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle \quad (t \in \mathbb{R}) \]

In both cases \( t \) is the *parameter*.

A line can also be written as a *symmetric equation*

\[ \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \]

A *line segment* is obtained by bounding \( t \) in the “parametric” formula to some closed interval. For example, the line segment from \( \mathbf{r}_0 \) to \( \mathbf{r}_1 \) is given by \( \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0) \) with \( 0 \leq t \leq 1 \). If two lines neither intersect nor are parallel, they are called *skew*.

**Planes.** There are several formulas/equations for planes. A plane through point \( \mathbf{r}_0 \) can be defined by requiring that the vector from \( \mathbf{r}_0 \) to any point in the plane must be orthogonal to a given direction \( \mathbf{n} \) called the *normal*:

\[ \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \]

If \( \mathbf{n} = \langle a, b, c \rangle \) and \( \mathbf{r}_0 = (x_0, y_0, z_0) \), then this is equivalent to

\[ a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \]

which is equivalent, for a suitable \( d \), to what is called the *symmetric equation*:

\[ ax + by + cz + d = 0 \]

To find the formula for a plane, one strategy is to start by finding a normal. For example, if we have two vectors in the plane, their cross product is a normal.

Planes are parallel if and only if they have the same normal. If two distinct planes intersect, they do so in a line; moreover a direction of that line is given by the cross product of the two planes' normals. The angle between two planes is equal to the angle between their normals.
12.6 Cylinders and Quadric Surfaces

In general, a **cylinder** is defined by a plane curve and consists of all lines perpendicular to the plane that pass through the curve.

The **trace** of a surface is its cross-section. **Quadric surfaces** have general formula

\[ Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0 \]

Quadric surfaces can be translated (meaning \( x' = x - x_0 \)) and rotated to standard forms of:

- **ellipsoid (football)**
  \[
  \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1
  \]

- **elliptic paraboloid (satellite-dish)**
  \[
  z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \quad \text{(axis is positive z-axis)}
  \]

- **hyperbolic paraboloid (saddle)**
  \[
  \frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}
  \]

- **hyperboloid (cooling-towers)**
  \[
  \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = \pm 1 \quad \text{(axis is z-axis)}
  \]

- **cone**
  \[
  \frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \quad \text{(axis is z-axis)}
  \]

Horizontal and vertical traces are **conic sections** (meaning ellipses, parabolas and hyperbolas). To get into standard form, start by completing the square.