Graphs Embedded on Surfaces

1 The Torus

The plane and the sphere are equivalent for embedding purposes. The simplest next surface is the torus (or donut). This can be thought of as a sphere with one hole in it, or a sphere with one handle added.

One can also flatten the handle to form a bridge: a rectangle where the left-to-right and top-to-bottom edges happen simultaneously. If we invert this picture, or if we cut a hollow torus, we see that: A torus can be obtained from a rectangle where the top and bottom are glued together, while the left and right are glued together. We say that the opposites sides are identified. Here is $K_7$ embedded on the torus, using this picture: the vertex at all four corners is the same vertex.

Now, it is important that there are two types of cycles. A contractible cycle can be continuously deformed or contracted to a single point. In the plane, all cycles are contractible, but on other surfaces they are not. In the above example, cycle 136 is contractible, but cycle 123 is noncontractible. If you slice through the torus on a noncontractible cycle, this will break the handle; if you slice through the torus on a contractible cycle, you will get two pieces, a sphere and a torus.

A region is called a 2-cell if its boundary is a contractible cycle. An embedding is a 2-cell embedding if every face is a 2-cell.

2 The Orientable Surface of Genus $h$

In general we can add $h$ handles. The resultant surface is called $S_h$ and $h$ is the genus of the surface. There is also a way to depict this as a $4h$-sided polygon with opposite sides suitably identified. There is an Euler’s formula:
Theorem 1  For a 2-cell embedding of a graph $G$ on $S_h$, it holds that $V - E + F = 2 - 2h$.

Proof. The proof is by induction on $h$. We already saw the result for $h = 0$.

So consider general $S_h$. Find a noncontractible cycle $C$ that does not go through a vertex of $G$. Since we have a 2-cell embedding, the cycle does not lie completely inside a region, and hence intersects some of the edges of $G$.

Now we can form a new graph $G'$ by (a) adding vertices at every intersection between $C$ and edges of $G$, and (b) adding edges embedded along $C$. The result is still a 2-cell embedding on the same surface.

Say there were $k$ intersections between $C$ and edges of $G$. Then $G'$ has $V + k$ vertices, $E + 2k$ edges ($k$ subdivisions and $k$ new edges) and $F + k$ faces.

Now, cut along the middle of $C$, thereby breaking the handle—called a capping operation. At the same time, duplicate each vertex and edge of $C$. The result is a graph $G''$ that has $V + 2k$ vertices, $E + 3k$ edges. Note that the cut exposes two new regions, bounded by the two copies of $C$. So the embedding has $F + k + 2$ faces, while it is on the surface with one less handle. It follows by induction that

$$(V + 2k) - (E + 3k) + (F + k + 2) = 2 - 2(h - 1).$$

And the result follows. QED

It follows that: $E \leq 3V + 6(h - 1)$.

That is, for any surface, if you have a sufficiently large graph then there is a vertex of degree at most 6.

3  The Genus of a Graph

The **genus** of a graph is the minimum number of handles for a surface needed to embed the graph.

Given each complete graph, one can apply Euler formula to get a lower bound on the genus. It is possibly surprising, but showing that the complete graph embeds on the surface it is supposed to takes a lot of proving! Finished by Ringel–Youngs.

In contrast, proving that the complete bipartite graph embeds on the lowest genus that Euler allows is not too hard. It is probably not too surprising that the genus of a graph is the sum of the genera of its blocks. (Battle, Harary, Kodama and Youngs).
4 Nonorientable Surfaces

The above surfaces are orientable. That means that there is a consistent definition of “clockwise” throughout the surface. In general a (closed) surface is a connected compact Hausdorff topological space which is local homeomorphic to an open disc.

Another possible addition is a crosscap. This is a circle which is twisted so that entering at one side comes out the opposite. The sphere with a single crosscap is the projective plane; with two it is the Klein bottle. This is nonorientable.

Then you can think of adding arbitrarily many crosscaps and handles. However, a standard fundamental result of algebraic geometry is that every such surface is equivalent to multiple handles or multiple crosscaps. This produces $S_h$ $h \geq 0$ and $N_k$ $k \geq 1$.

Euler formula for $N_k$: $V - E + F = 2 - k$. 