Algorithms/Complexity on Planar Graphs

1 Divide and Conquer

Lipton and Tarjan proved a planar separator theorem. This can be used for efficient divide and conquer algorithms. Miller proved a cycle separator theorem. We will only state their result, but we prove some simple separator theorems.

Lemma 1 If $T$ is a tree, there is a vertex $v$ such that $T - v$ has maximum component size at most $n/2$.

Proof. Root the tree at any vertex. If the largest component below the root is bigger than $n/2$, then move to that child. If the largest component below still exceeds $n/2$, then move to its biggest child. Since the tree is finite, there must come a time we move to a child and the largest component below is small enough. Stop: that is the right vertex. QED

Theorem 2 (Leighton) If $G$ is maximal outerplanar, then there is a triangle $v$ whose removal leaves maximum component size at most $n/2$.

Proof. Apply the previous lemma to the weak dual. QED

We state the separator theorem:

Theorem 3 (Lipton & Tarjan) If $G$ is a planar graph, then one can partition the vertices of $G$ into 3 sets $A$, $B$ and $C$ such that there is no edge between $A$ and $C$, and $|B|$ is $O(\sqrt{n})$.

Proof. The preliminary step is to show we may assume that the graph has a spanning tree of radius $O(\sqrt{n})$. To see this, take any spanning tree $T$. Then for $1 \leq i \leq \sqrt{n}$, consider the set $C_i$ of vertices at distance exactly $i + a\sqrt{n}$ for all $a$. By picking the smallest such set, we can remove $\sqrt{n}$ vertices so that all pieces have spanning tree of radius at most $\sqrt{n}$.

The main part of the proof considers the tree $T$ of small radius. Consider the edges of the dual that do not cross $T$: that is a tree! Then do some elegant manipulation of weights, and finally appeal to the tree separator theorem. QED

By repeated use of (a weighted version) of this, one can find $\sqrt{n}/\varepsilon$ vertices whose removal leaves every component of size at most $\varepsilon n$. 
**Application.** We have a $1 - o(1)$ approximation algorithm for the maximum independent set. Repeatedly use above (or a generalization thereof) to remove $\alpha$ vertices such that every component that remains has size at most $\beta$. Use exhaustive search to find a maximum independent set in each component, and take the union of the results as the approximation. $\beta$ has to be very small for exhaustive search to work—certainly at least below $O(\log n)$. The above theorem guarantees that what is lost is small relative to the actual answer. (Note that we know the real answer is at least $n/4$, by the 4-color theorem.)

2 NP-completeness for Planar Graph Problems

Often one wants to show that a graph problem is NP-complete even restricted to planar graphs.

Approach one: reduce from the general problem to the planar one. The reduction is to produce a planar gadget $F$ so that when one draws any graph in the plane, replacing each crossing by an $F$ preserves the parameter in some sense.

For example, Garey, Johnson, Stockmeyer showed that Vertex Cover was NP-hard even for planar graphs by reducing from normal Vertex Cover. Each crossover was replaced by the following gadget:

The original edges come in at $a$, $a'$, $b$, $b'$. The gadget has certain properties about the minimum size of a vertex cover of it given how many of each outlets are in the set. The proof is left as an exercise.

Approach two: Lichtenstein offers a generic approach. He showed that Planar 3SAT it NP-complete. That is, 3SAT is NP-complete even when restricted to the case that the natural variable–clause incidence graph (together with a cycle through the variables in order) is planar.