Minors and Well-quasi-orderings

1 The Trees are Well-quasi-ordered by Topological Minor

A quasi-ordering is a reflexive transitive relation. A set $X$ is well-quasi-ordered by $\leq$ if given any infinite subsequence $\{x_k\}$ of $X$ there are two elements $x_i$ and $x_j$ with $i < j$ and $x_i \leq x_j$.

Lemma 1 A set is WQO iff it contains neither an infinite antichain (no two elements comparable) nor an infinite strictly decreasing sequence.

Proof is left as an exercise.

For two subsets $A$ and $B$ of $X$, we define $A \leq B$ if there is a 1–1 mapping $f$ from $A$ to $B$ such that $a \leq f(a)$ for all $a \in A$.

Lemma 2 If $X$ is WQO by $\leq$, then so is the set of finite subsets of $X$.

Proof omitted.

We say that graph $G$ is a topological minor of $H$ is $H$ contains a subgraph that is a subdivision of $G$. People also say that $G$ is a homeomorphic subgraph of $H$.

Theorem 3 (Kruskal) The finite trees are WQO by the topological minor relation.

That is, given any infinite sequence of trees, there is $i < j$ such that tree $T_j$ contains a subdivision of tree $T_i$ as a subgraph.

Proof. Consider rooted trees. Suppose the result is false. That is, we can find an infinite sequence of rooted trees such that none is a topological minor of a later one. Choose such a sequence one tree at a time by letting the tree $T_n$ be a minimum-order rooted tree such that there is still an infinite bad sequence starting $T_1, \ldots, T_n$.

Let $C_n$ be the multiset of child subtrees of tree $T_n$. Let $A$ be the union of the $C_n$. We show that $A$ is WQO. For consider any infinite sequence $B_1, B_2, \ldots$ of $A$. Find the smallest $j$ such that there is a $B_i \in C_j$. And consider the sequence $T_1, T_2, \ldots, T_{j-1}, B_i, B_{i+1}, \ldots$.

By the choice of the sequence $\{T_n\}$, and that $B_i$ is part of $T_j$, it follows that the new sequence is not bad. So there are two trees in it, with one a topological minor of the later. These two trees cannot both be in the $T_1 \ldots T_{j-1}$ portion, because the original sequence is bad. Nor in fact can the first tree be from that portion,
because being a topological minor of $B_k$ implies it is a topological minor of the tree that $B_k$ is a child in. So the two trees are in the $B_i$ portion, as required. That is, $A$ is WQO.

It follows from earlier lemma that the finite subsets of $A$ are also WQO by topological minor. In particular, we can always find $i$ and $j$ so that the multiset $C_i$ of child-trees of $T_i$ is $\leq$ the multiset $C_j$ of child-trees of $T_j$. That is, one can match each tree of $C_i$ with a tree of $C_j$ such that the first is a topological minor of the second. It follows that $T_i$ is a topological minor of $T_j$. Which is a contradiction. QED

2 The Minor Theorems

Theorem 4 (Robertson & Seymour) The finite graphs are WQO by the minor relation.

Corollary 5 Every graph property closed under minors has a characterization by a finite list of forbidden minors.

In particular, for every surface there is a finite list of forbidden minors. Actually, they proved this consequence as a stepping stone to the overall theorem.

There is an $O(n^3)$ algorithm to test for existence of fixed minor.

For example, a linkless embedding of a graph is an embedding of the graph in 3 dimensions such that there are no two interlocking cycles. Having a linkless embedding is clearly closed under taking minors. So before the problem was not known to be decidable; now it is known to be in P.

But note that the algorithms have huge hidden constants, and are nonconstructive in parts.