3 Properties of Binomial Coefficients

3.1 Properties of Binomial Coefficients

Here is the famous recursive formula for binomial coefficients. For $1 \leq k < n$,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$ 

This equation can be proven by replacing each binomial coefficient by its ratio of factorials and checking that we get the same on both sides. (Do it!)

However, mathematicians like proofs that explain why something is true: a combinatorial proof of an equation is where both sides are shown to count the same thing. In the above equation, the LHS (left-hand side) by definition counts the unordered subsets of size $k$. Now, let $a$ be the first element of the universe. A subset either contains $a$ or it doesn’t. If the subset contains $a$, then what remains is a subset of size $k-1$ from the remaining universe of size $n-1$. If the subset does not contain $a$, then it is a subset of size $k$ from the remaining universe of size $n-1$. So by the sum rule, the RHS (right-hand side) also counts the unordered subsets of size $k$: the first binomial coefficient counts those with $a$ and the second binomial coefficient counts those without.

Binomial coefficients are also found in Pascal’s triangle. Pascal’s triangle has the rule that each entry is the sum of the two entries just above it, and so the $n^{th}$ row from the top is the binomial coefficients $\binom{n}{k}$. Many thousands of pages have been written about the properties of binomial coefficients and their kin.

For example, the remainders when binomial coefficients are divided by a prime provide interesting patterns. Here is the start of Pascal’s triangle with the odd binomial coefficients shaded.
Here is another famous fact about binomial coefficients.

**Theorem 3.1** For \( n \geq 0 \),

\[
\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.
\]

**Proof.** We give a combinatorial proof. Let \( X \) be an \( n \)-element set, and let \( Y \) be the set of subsets of \( X \). In Example 1.5 we observed that \( |Y| = 2^n \).

On the other hand, if we count \( Y \) by considering the sizes of each subset, by Lemma 2.1 there are \( \binom{n}{k} \) of size \( k \), and so, if we sum this quantity from \( k = 0 \) to \( k = n \), we get \( |Y| \). Thus the two sides of the above equation are in fact equal. ◊

If we use sigma-notation, then the above equation can be rewritten as

\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n.
\]

In the expression \( \sum_{k=0}^{n} \), it means to loop through all values from \( k = 0 \) to \( k = n \), evaluate the formula, and add up all the results.
The result in the previous theorem is generalized in the famous Binomial Theorem. (It’s a generalization, because if we plug \( x = y = 1 \) into the Binomial Theorem, we get the previous result.)

**Theorem 3.2 (Binomial Theorem)**

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.
\]

**Proof.** Let’s start by showing the idea in a specific case. Consider \( n = 3 \). Then the LHS product is \((x + y)(x + y)(x + y)\). If we multiply this out, but do not use the commutative law for multiplication, we get \(xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy\). Now, to get the coefficient of \(x^2y\) say, we group together \(xxy\), \(xyx\), and \(yxx\). That is, the coefficient of \(x^2y\) is the number of ways of creating a “word” using exactly \(x\), \(x\), and \(y\). To count such, we choose the positions for the \(y\)’s: this is a subset of size \(k\).

The real proof is exactly the above idea but with notation. The total number of \(x^{n-k}y^k\) in \((x + y)^n\) is equal to the number of ways of placing the \(k\) \(y\)’s in a word together with \(n - k\) \(x\)’s; this is the binomial coefficient \(\binom{n}{k}\). ◊

**Exercises**

3.1. Provide a combinatorial proof of the identity:

\[
n \binom{n-1}{2} = \binom{n}{2} (n-2)
\]

(Hint: Consider a three-person subcommittee.)

3.2. Show that if \(p\) is a prime number, then \(\binom{p}{i}\) is a multiple of \(p\) for all \(i\) between 1 and \(p - 1\).

3.3. Consider the following identity:

\[
\sum_{0 \leq i \leq n/2} \binom{n}{2i} = \sum_{0 \leq i < n/2} \binom{n}{2i + 1}.
\]

For example, when \(n = 3\) it claims that \(\binom{3}{0} + \binom{3}{2} = \binom{3}{1} + \binom{3}{3}\), which is true since both sides equal 4.

(a) Verify this identity for \(n = 4\) and \(n = 5\).

(b) Deduce this identity from the Binomial Theorem (by plugging in suitable value of \(x\) and \(y\)).
(c) Give a combinatorial proof of the identity.

3.4. Consider the following identity:

\[ \sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}. \]

For example, when \( n = 2 \) it claims that \( \binom{2}{0}^2 + \binom{2}{1}^2 + \binom{2}{2}^2 = \binom{4}{2} \), which is true since both sides equal 6.

(a) Verify this identity for \( n = 3 \) and \( n = 4 \).
(b) Give a combinatorial proof of the identity. (Hint: consider a \( 2n \)-element set with half its elements colored red.)

3.5. Just like \( \sum \) for addition, there is \( \prod \) for multiplication. Show that

\[ \binom{n}{k} = \prod_{i=1}^{k} \frac{n-k+i}{i} \]

3.6. (a) Using some mathematics software or a calculator, calculate \( \binom{50}{25} \).
(b) In Java (and usually in C) an int variable has a maximum value of \( 2^{31} \). Explain why we cannot use int’s to calculate 50!.
(c) Write code using the recursive formula to calculate \( \binom{50}{25} \). (Note that you will need to stop the recursion under certain circumstances such as when the top or bottom part of the binomial coefficient becomes zero.)
(d) Write code using the formula from Exercise 3.5 to calculate \( \binom{50}{25} \).
(e) Comment on the efficiency of your code.

3.7. Suggest and prove a generalization of the Binomial Theorem of the form \((x+y+z)^n = \sum \ldots\).