Location Functions on Trees: A Survey and Some Recent Results

F.R. McMorris

Department of Applied Mathematics
Illinois Institute of Technology
Chicago, Illinois
&
Department of Mathematics
University of Louisville
Louisville, Kentucky
“Working with trees

“Working with trees is just monkey business”
“During the past 15 years, this area of network location has experienced a fairly rapid growth – in terms of both potential applications and theoretical development.” – D.R. Shier & P.M Dearing, *Operations Research* 1983
Outline of the talk

- Introduction
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- Three standard location functions on finite metric spaces
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- Properties ("axioms") used to characterize these functions
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- Properties ("axioms") used to characterize these functions
- Some old and very new results
The general setting

Let \((X, d)\) be a finite metric space and \(X^* = \bigcup_{k>1} X^k\). The elements of \(X^*\) are called *profiles* and are denoted \(\pi = (x_1, \ldots, x_k)\), \(\pi' = (y_1, \ldots, y_m)\), etc.
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A location function on \(X\) is be a function of the form \(L : X^* \rightarrow 2^X \setminus \{\emptyset\}\). Our interest is in location functions that return, for any profile \(\pi\), a set of points that minimize an objective criterion of “remoteness” from \(\pi\).
Location functions on metric space \((X, d)\)

Locating a fire station versus locating a mall or distribution center.

\[ \text{Median Function: } Med(\pi) = \{ x \in X : \sum_{i=1}^{k} d(x, x_i) \text{ is minimum} \} \]

\[ \text{Center Function: } Cen(\pi) = \{ x \in X : e(x, \pi) \text{ is minimum} \} \]

\[ e(x, \pi) = \max\{ d(x, x_1), d(x, x_2), \ldots, d(x, x_k) \} \]

\[ \text{Mean Function: } \text{Mean}(\pi) = \{ x \in X : \sum_{i=1}^{k} d^2(x, x_i) \text{ is minimum} \} \]
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The mean and median functions are special instances of the following location function that is inspired by the $\ell_p$-norm $\| \cdot \|_p$ with $\| \pi \|_p = \sqrt[p]{\sum_{i=1}^{k} d^p(x, x_i)}$.

$\ell_p$-function: $\ell_p(\pi) = \{ x \in X : \sum_{i=1}^{k} d^p(x, x_i) \text{ is minimum} \}$, for any profile $\pi$. 

$\ell_p$-function
Properties of \textit{Med}

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Betweenness (B): $Med((x, y)) = \{z : d(x, y) = d(x, z) + d(z, y)\}$. 
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Betweenness (B): \( \text{Med}((x, y)) = \{z : d(x, y) = d(x, z) + d(z, y)\} \).

Consistency (C): If \( \text{Med}(\pi_1) \cap \text{Med}(\pi_2) \neq \emptyset \) for profiles \( \pi_1 \) and \( \pi_2 \), then \( \text{Med}(\pi_1 \pi_2) = \text{Med}(\pi_1) \cap \text{Med}(\pi_2) \), where \( \pi_1 \pi_2 \) is the concatenation of \( \pi_1 \) and \( \pi_2 \), i.e., if \( \pi_1 = (x_1, \ldots, x_k) \) and \( \pi_2 = (y_1, \ldots, y_m) \) then \( \pi_1 \pi_2 = (x_1, \ldots, x_k, y_1, \ldots, y_m) \).
Proof of (C)

Let $D(x, \pi) = \sum_{i=1}^{k} d(x, x_i)$.

Assume $\text{Med}(\pi_1) \cap \text{Med}(\pi_2) \neq \emptyset$. Let $x \in \text{Med}(\pi_1) \cap \text{Med}(\pi_2)$, and $z \in \text{Med}(\pi_1 \pi_2)$.

Then $D(x, \pi_1 \pi_2) = D(x, \pi_1) + D(x, \pi_2) \leq D(z, \pi_1) + D(z, \pi_2) = D(z, \pi_1 \pi_2) \leq D(x, \pi_1 \pi_2)$, so $x \in \text{Med}(\pi_1 \pi_2)$.

From above, $D(x, \pi_1) + D(x, \pi_2) = D(z, \pi_1) + D(z, \pi_2)$ so $D(x, \pi_1) \leq D(z, \pi_1)$ and $D(x, \pi_2) \leq D(z, \pi_2)$ imply $D(x, \pi_1) = D(z, \pi_1)$ and $D(x, \pi_2) = D(z, \pi_2)$. i.e., $z \in \text{Med}(\pi_1) \cap \text{Med}(\pi_2)$. 

Axioms for location function

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Answer: Sometimes “yes”!
ABC Theorem

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Note that the points in this space are $V$ and that all location points must be on the vertices $G$. (There is a large literature concerned with location on networks where edges have lengths and location points can be placed on the edges.)
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A more general result in (McM, H.M. Mulder, F.S. Roberts, DAM 1998) gives the

**Collorary:** Let $L$ be a location function on the tree $T$. Then $L = Med$ if and only if $L$ satisfies (A), (B) and (C).
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The ABC saga over more than a decade resulted in the best ABC theorem to date (H.M. Mulder & B. Novick, DAM 2012?):

**Theorem** Let $L$ be a location function on a median graph $G$. Then $L = Med$ on $G$ if and only if $L$ satisfies (A), (B) and (C).
Bob Powers found an example on the covering graph of the 5 element non-distributive lattice $M_3$ (the “diamond lattice”) where the ABC Theorem does not hold. i.e., there exists a location function that is not the median function, but does satisfy the three axioms.
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**Open Problem:** Characterize those graphs where the ABC theorem is true.
The Mean Function

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Recall that the mean function is defined by

$$\text{Mean}(\pi) = \{ x \in X : \sum_{i=1}^{k} d^2(x, x_i) \text{ is minimum} \}.$$
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$\text{Mean}$ does not satisfy (B), but does satisfy the analogous axiom (J. Biagi, UofL MA Thesis, 2000):

**Middleness (Mid):** Let $x, y \in V$. If $d(x, y)$ is even, then $L((x, y)) = K_1$ where $d(x, K_1) = d(y, K_1) = d(x, y)/2$. If $d(x, y)$ is odd, then $L((x, y)) = K_2$ where $d(x, K_2) = d(y, K_2)$. 
**Conjecture:** Let $L$ be a location function on a tree $T$. Then $L = \text{Mean}$ if and only if $L$ satisfies (A), (Mid) and (C).
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Unfortunately \textbf{ALL} the $\ell_p$-functions for $p > 1$ satisfy these three axioms.
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We needed to add a strong “Property Z” in (McM, H.M. Mulder, O. Ortega, DMAA 2010) to get

Theorem: Let $L$ be a location function on a tree $T$. Then $L = \text{Mean}$ if and only if $L$ satisfies (A), (Mid), (C) and (Z).
Property Z

\[ R_\pi(a, b) = \sum_{x \in \pi_{ba}} d(b, x) - \sum_{x \in \pi_{ab}} d(b, x), \]
\[ D_\pi(a, b, c) = 2 \sum_{x \in \pi_{abc}} d(b, x). \]

Property (Z) : Let \( \pi = (x_1, x_2, \ldots, x_n) \) be a profile and \( L \) be a location function on \( T \). If \( L(\pi) = \{a\} \), then
\[ R_\pi(b, a) + R_\pi(c, b) > D_\pi(a, b, c) \]
whenever \( ab, bc \in E \).
In another direction

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For a profile $\pi = (x_1, \ldots, x_k)$ and vertex $z$, let $\pi \diamond z = (x_1, \ldots, x_k, z)$.

**Invariance** (In): Let $L$ be a location function on a tree $T$. $L$ satisfies (In) if: $a \in L(\pi)$ if and only if $L(\pi \diamond a) = \{a\}$ for any profile $\pi$. 
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It can be shown with some work that *Mean* satisfies (In); and also the following:

**Unanimity** (U): For any vertex $a \in T$, $L((a, a, \ldots, a)) = \{a\}$. 
In (McM, H.M. Mulder, O. Ortega, Networks 2012) we showed:

**Theorem**: If $L$ is a location function on the tree $T$ and $p > 1$, then $L = ℓ_p$ if and only if $L$ satisfies (In), (U), and “p-Projection”.
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**Theorem**: If $L$ is a location function on the tree $T$ and $p > 1$, then $L = \ell_p$ if and only if $L$ satisfies (In), (U), and “p-Projection”. (p-Projection is another technical axiom that does the heavy lifting)

**Theorem**: If $L$ is a location function on the tree $T$, then $L = \text{Mean}$ if and only if $L$ satisfies (In), (U), and “2-Projection”.
**p-Projectiveness (p-P):** Let $\pi = (x_1, x_2, \ldots, x_k)$ be a profile of odd length, let $L$ be a location function on $T$, and let $p$ be an integer with $p > 1$. Let $\beta = (z_1, z_2, \ldots, z_k)$ be an out-projected profile of $\pi$, and assume $L(\pi) = \{a\}$. If

$$
0 < \binom{p}{1} \ell_{p-1} S_{\beta}(a) + \cdots + \binom{p}{p-1} \ell_{1} S_{\beta}(a) - 2 \binom{p}{1} \ell_{p-1} S_{\beta_{ba}}(a) - 2 \binom{p}{3} \ell_{p-3} S_{\beta_{ba}}(a) - \cdots - 2 \binom{p}{p-1} \ell_{1} S_{\beta_{ba}}(a) + |\pi_{ab}| + |\pi_{ba}| (-1)^p
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for any vertex $b$ adjacent to $a$, then $L(\beta) = \{a\}$. 

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**Open Problems:** Find properties much easier to grasp than (Z) and (p-P) that allow for a characterization of Mean on trees. Extending beyond trees is open.
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Center Function on Trees

Recall the definition: \( \text{Cen}(\pi) = \{ x \in X : e(x, \pi) \text{ is minimum} \} \), where \( e(x, \pi) = \max\{d(x, x_1), d(x, x_2), \ldots, d(x, x_k)\} \).
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On trees, \( Cen \) clearly satisfies (Mid), and hence not (B) and the
following example shows that it does not satisfy (C):

\[
\text{Let } T \text{ be the path } x_1 x_2 x_3 x_4, \pi_1 = (x_1, x_4) \text{ and } \pi_2 = (x_1, x_3).
\text{Then } Cen(\pi_1) = \{x_2, x_3\} \text{ and } Cen(\pi_2) = \{x_2\} \text{ which gives}
Cen(\pi_1) \cap Cen(\pi_2) = \{x_2\}.
\text{But } Cen((x_1, x_4, x_1, x_3)) = \{x_2, x_3\}.
\text{But } Cen \text{ does satisfy (on any finite metric space):}
\text{Quasi-Consistency (QC): If } L(\pi) = L(\pi') \text{ for profiles } \pi \text{ and } \pi',
\text{then } L(\pi \pi') = L(\pi). \text{Clearly if a location function } L \text{ satisfies (C) then it satisfies (QC).}
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Let \( T \) be the path \( x_1x_2x_3x_4 \), \( \pi_1 = (x_1, x_4) \) and \( \pi_2 = (x_1, x_3) \). Then \( \text{Cen}(\pi_1) = \{ x_2, x_3 \} \) and \( \text{Cen}(\pi_2) = \{ x_2 \} \) which gives \( \text{Cen}(\pi_1) \cap \text{Cen}(\pi_2) = \{ x_2 \} \). But \( \text{Cen}((x_1, x_4, x_1, x_3)) = \{ x_2, x_3 \} \).
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But \( \text{Cen} \) does satisfy (on any finite metric space):

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then \( L(\pi\pi') = L(\pi) \).
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Recall the definition: \( \text{Cen}(\pi) = \{ x \in X : e(x, \pi) \text{ is minimum} \} \), where \( e(x, \pi) = \max\{ d(x, x_1), d(x, x_2), \ldots, d(x, x_k) \} \).

On trees, \( \text{Cen} \) clearly satisfies (Mid), and hence not (B) and the following example shows that it does not satisfy (C):

Let \( T \) be the path \( x_1 x_2 x_3 x_4 \), \( \pi_1 = (x_1, x_4) \) and \( \pi_2 = (x_1, x_3) \). Then \( \text{Cen}(\pi_1) = \{ x_2, x_3 \} \) and \( \text{Cen}(\pi_2) = \{ x_2 \} \) which gives \( \text{Cen}(\pi_1) \cap \text{Cen}(\pi_2) = \{ x_2 \} \). But \( \text{Cen}((x_1, x_4, x_1, x_3)) = \{ x_2, x_3 \} \).

But \( \text{Cen} \) does satisfy (on any finite metric space):

**Quasi-Consistency (QC):** If \( L(\pi) = L(\pi') \) for profiles \( \pi \) and \( \pi' \), then \( L(\pi \pi') = L(\pi) \).

Clearly if a location function \( L \) satisfies (C) then it satisfies (QC).
Characterization of $Cen$

For a profile $\pi$, let $\{\pi\}$ be the set of elements making up $\pi$. Then $Cen$ clearly satisfies the next property (on any finite metric space) for a location function $L$.
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Final axiom:

**Redundancy** (R): Let $L$ be a location function on a tree $T$. If $x \in T(\{\pi\} - x)$, then $L(\pi - x) = L(\pi)$.
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Characterization of \( Cen \)

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**Theorem**: Let \( L \) be a location function on a tree \( T \). Then \( L \) is the center function \( Cen \) on \( T \) if and only if \( L \) satisfies properties \((Mid)\), \((PI)\), \((QC)\) and \((R)\).
Characterize $Cen$ on a more general class of graphs than trees.
Characterize $\text{Cen}$ on a more general class of graphs than trees.

Results needed for 2-median, 2-mean, 2-center, etc.