On the Maximum Degree of $3_t$-Critical Graphs

Francesco Barioli    Lucas van der Merwe$^1$

Department of Mathematics
University of Tennessee at Chattanooga

$^1$Research supported by Math. Depart. UTC
Outline

1 Introduction
2 Background
3 General Results
   - General Bounds
4 Sharp Upper and Lower Bounds
   - $\alpha(G) = 2$
   - Crown Graphs
   - Claw-Free Graphs
5 Summary
\( N(u) \) is the open neighborhood of a vertex \( u \) in a graph, whereas \( N[u] = N(u) \cup \{u\} \) is the closed neighborhood of \( u \).

The degree of a vertex \( u \) is \( |N(u)| \) denoted \( \text{deg}(u) \).

The minimum degree and maximum degree of \( G \) are defined by

\[
\delta(G) = \min_{v \in V(G)} \text{deg}(v), \quad \Delta(G) = \max_{v \in V(G)} \text{deg}(v),
\]

respectively.
$S$ is a **total dominating set** of $G$ if every vertex of $V(G)$ is adjacent to some vertex of $S$. We write $S \succ_t G$. (First introduced by Cockayne, Dawes and Hedetniemi, (1980)).

The minimum cardinality of $S$ is the **total domination number** of $G$ denoted by $\gamma_t(G)$.

$S$ is a **dominating set** of $V(G)$ if every vertex of $V(G) - S$ is adjacent to some vertex of $S$.

The minimum cardinality of $S$ is the **domination number** of $G$ denoted by $\gamma(G)$.
The total domination number deceases by at most two when adding an edge to a graph $G$. That is,

$$\gamma_t(G) - 2 \leq \gamma_t(G + e) \leq \gamma_t(G)$$

Graphs for which $\gamma_t(G + e) = \gamma_t(G) - 2$ for each $e \notin E(G)$ are called supercritical.
If for three given vertices $u, v, w$ we have $\{u, v\} \succ_{t} V(G) - w$, we write $uv \rightarrow w$.

**Theorem 1**

Let $\gamma_t(G) = 3$. Then is $G$ is $3_t$-critical if and only if for any pair of non adjacent vertices $u$ and $v$, either

1. $\{u, v\} \succ V(G)$, or
2. $uw \rightarrow v$ for some $w \in N(u)$, or
3. $vw \rightarrow u$ for some $w \in N(v)$. 
\{1, 4, 5\} is a $\gamma_t$-set among others. Now add an edge $e \in \overline{G}$.
A graph $G$ is diameter 2-critical if $\text{diam}(G) = 2$ and $\text{diam}(G - e) > 2$ for any $e \in E(G)$.

Hanson and Wang showed the following important relationship.

**Theorem 2**

* A graph is diameter 2-critical if and only if its complement is either $3_t$-critical or $4_t$-supercritical.
Murty and Simon posed a conjecture on diameter 2-critical graphs, namely:

**Conjecture 3**

If $G$ is a diameter 2-critical graph with order $n$ and size $m$, then \( m \leq \lceil n^2/4 \rceil \), with equality if and only if $G$ is the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

**Theorem 4**

If $G$ be a graph with order $n$ and size $m$, then
1. if $G$ is $3_t$-critical, then $m > \lceil n(n - 2)/4 \rceil$;
2. if $G$ is $4_t$-supercritical, then $m \geq \lceil n(n - 2)/4 \rceil$.

Condition (2) has been settled by Hanson and Wang (2003).
Recently, Condition (1) was settled for the two following families of $3_t$-critical graphs. (Henning, Haynes, Yeo, vdM)

**Theorem 5**

*If $G$ is a $3_t$-critical graph of diameter 3, order $n$, and size $m$, then $m > \lceil n(n - 2)/4 \rceil$.***

**Theorem 6**

*If $G$ is a $3_t$-critical claw-free graph of order $n$ and size $m$, then $m > \lceil n(n - 2)/4 \rceil$.***
**Theorem 7**

Let $G$ be a $3_t$-critical graph. Then $|G| \geq 5$. Furthermore, if $|G| = 5$ then $G = C_5$, the cycle on five vertices.

**Theorem 8**

(Cockayne et al.) If a graph $G$ is connected and $\Delta(G) < |G| - 1$, then $\gamma_t(G) \leq |G| - \Delta(G)$.

**Observation 9**

Any graph $G$ with $\gamma_t(G) = 3$ has $\Delta(G) \leq |G| - 3$. More generally, $\Delta(G) \leq |G| - \gamma_t(G)$. 
Theorem 10

*The only* $3_t$-*critical* $|G|−3$-*regular graph $G$ is $C_5$.*

Theorem 11

*Let $G$ be a $3_t$-*critical* graph with $|G| > 5$. Then* $\Delta(G) \geq \lceil \frac{1}{2} |G| \rceil$.

Combining Observation 9 and Theorem 11, gives:

Theorem 12

*If* $G$ *is a* $3_t$-*critical graph, then* $\lceil \frac{1}{2} |G| \rceil \leq \Delta(G) \leq |G| − 3$.  

Barioli, van der Merwe  Max Degree
The *independence number* $\alpha(G)$ of a graph $G$ is defined as the maximum number of pairwise disjoint vertices of $G$.

**Lemma 13**

If $\gamma_t(G) = 3$ and $\alpha(G) = 2$, then $G$ is $3_t$-critical.

Another straightforward yet important fact:

**Lemma 14**

Let $G$ be a $3_t$-critical graph with $\alpha(G) = 2$. Then two vertices are adjacent if and only if they share a common non-neighbor.

In particular, if $u$ and $v$ are non-adjacent vertices, we have

$$|G| = \deg(u) + \deg(v) - |N(u) \cap N(v)| + 2.$$  \hfill (1)
Theorem 15

Let $G$ be a $3_t$-critical graph with $\alpha(G) = 2$. Then $\Delta(G) \geq \lceil \frac{3}{5} |G| - 1 \rceil$.

Outline of proof.

1. Let $n = |G|$, $k = \Delta(G)$.
2. Among all pairs of nonadjacent vertices in $G$, let $(u, v)$ be a pair maximizing $\deg(u) + \deg(v)$.
3. $\deg(u) + \deg(v)$ is maximum when $|N(u) \cap N(v)|$ is maximum.
4. Partition $V(G) \setminus \{u, v\}$ into the three subsets

$$A = N(u) \setminus N(v), \quad B = N(v) \setminus N(u), \quad C = N(u) \cap N(v).$$
\[ |A| + |C| = \deg(u) \leq k; \quad |B| + |C| = \deg(v) \leq k, \quad (2) \]

- By Lemma 14, both \( A \) and \( B \) are complete.
\( \alpha(G) = 2 \)

Crown Graphs
Claw-Free Graphs

\[ C \text{ is necessarily non-empty. } \{c, v\} \succ V(G) \setminus A, \text{ while } \{c, v\} \not\succ V(G). \] 
There exists \( a \in A \) that is not adjacent to \( c \).
Similarly, \( b \in B \) nonadjacent to \( c \).
\( \{a, c\} \succ G \) and \( \{b, c\} \succ G \). By the maximality of \( \deg(u) + \deg(v) \), \( \deg(a) \leq \deg(u) \). Since \( a \succ A \setminus \{a_1\} \), we have

\[
|N_B(a)| = \deg(a) - (|A| - 1) - 1 - |N_C(a)|
\]

\[
\leq \deg(a) - |A| \leq \deg(u) - |A| = |C|.
\]
Since $v \in N(b) \cap N(c)$, we have $N_B(c) \subset N(b) \cap N(c)$, so that

$$|N_B(c)| < |N(b) \cap N(c)| \leq |N(u) \cap N(v)| = |C|.$$
Finally, since \( \{a, c\} \not
CREASE B, we have

\[
|B| \leq |N_B(a)| + |N_B(c)| \leq 2|C| - 1. \tag{3}
\]

From (2) and (3) we then obtain

\[
n = |A| + |B| + |C| + 2
= \frac{3(|A| + |C|) + 2(|B| + |C|) + |B| - 2|C| + 6}{3}
\leq \frac{5k + 5}{3} = \frac{5}{3}(k + 1),
\]

which completes the proof.
Vertices $u$ and $v$ of $G$ are *twin vertices* (or *duplicate vertices*) if $N[u] = N[v]$. A graph $G'$ is said to be obtainable from $G$ by *vertex duplication* if $G'$ has twin vertices $u$ and $v$ such that $G = G' - \{v\}$.

**Lemma 16**

Let $G$ be a graph with $\alpha(G) = 2$, and let $G'$ be obtainable from $G$ by vertex duplication. Then

1. $\alpha(G') = 2$;
2. $G'$ is $3_t$-critical if and only if $G$ is $3_t$-critical.

It is important to observe that a similar result does not hold when $\alpha(G) > 2$. 
A graph $G$ is said to be a *5-cycle-type graph* if $G$ can be obtained from the 5-cycle $C_5$ through a sequence of vertex duplications. In particular, if the sizes of these cliques are $c_1, \ldots, c_5$, we will write $G = C(i_1, i_2, i_3, i_4, i_5)$.

**Figure:** The graph $C(3, 1, 2, 2, 1)$. 

\[
\alpha(G) = 2
\]

Crown Graphs
Claw-Free Graphs
Proposition 17

Let $G = C(i_1, i_2, i_3, i_4, i_5)$ be a 5-cycle-type graph. Then

i. $|G| = \sum_{j=1}^{5} i_j$,

ii. $\Delta(G) = \max_j (i_{j-1} + i_j + i_{j+1}) - 1$,

iii. $\alpha(G) = 2$,

iv. $G$ is $3_t$-critical,

where indices have to be considered modulo 5.
Theorem 18

Let \( n, k \in \mathbb{N} \) such that \( n \geq 5 \) and \( \left\lceil \frac{3n}{5} - 1 \right\rceil \leq k \leq n - 3 \). Then there exists a \( 3_t \)-critical graph \( G \) with \( \alpha(G) = 2 \), \( |G| = n \), and \( \Delta(G) = k \).

Proof. Using Proposition 17, it is easy to show that \( G = C(n - 4p, \lceil p \rceil, \lceil p \rceil, \lceil p \rceil, \lceil p \rceil) \) where \( p = \frac{n-k-1}{2} \) satisfies all of the requirements.
Another interesting property of 5-cycle-type graphs is that they exhaust the class of $3_t$-critical graphs with $\alpha(G) = 2$ and $\Delta(G) = |G| - 3$, as stated in the following theorem.

**Theorem 19**

*Let $G$ be a $3_t$-critical graph with $\alpha(G) = 2$ and $\Delta(G) = |G| - 3$. Then $G$ is a 5-cycle-type graph.*
Another interesting class of $3_t$-critical graphs with $\alpha(G) = 2$, is the subclass of circulant graphs $C(n) = Ci_n \left( 1, 2, \ldots, \frac{n-2}{3} \right)$, for any $n \equiv 2 \pmod{3}$.

We just observe that, since $C(5) = C_5$, we can extend this class of circulant graphs by vertex duplication to contain the class of 5-cycle-type graphs. However, there are still examples of $3_t$-critical graphs $G$ with $\alpha(G) = 2$ that cannot be obtained from any circulant graph of the form $C(n)$ through vertex duplication, as, for instance, the complement of the Petersen graph.
$\alpha(G) = 2$

Crown Graphs

Claw-Free Graphs

Figure: The graphs $C_8$ and the complement of Petersen graph
A graph $G$ is said to be a *Crown graph* if $G$ is $3_t$-critical, and, for any pair of nonadjacent vertices $u$ and $v$ in $G$, there exist vertices $x$ and $y$ in $G$ such that $ux \rightarrow v$ and $vy \rightarrow u$.
We note that if $G$ is a $3_t$-critical graph with nonadjacent vertices $u$ and $w$ in $G$, and $uv \leftrightarrow w$, for some vertex $v$, then we have

$$|G| = \deg(u) + \deg(v) - |N(u) \cap N(v)| + 1. \quad (4)$$
Lemma 20

Let $G$ be a Crown graph. Then $|G| \leq \Delta(G) + \frac{1}{2} \delta(G) + 2$.

Proof.

- Let $k = \Delta(G)$, $\delta = \delta(G)$, and let $u$ be a vertex of degree $\delta$. Let $A = N(u)$, and let $B = V(G) - (A \cup \{u\})$.
- Then $|B| = n - \delta - 1$ with $B = \{b_1, \ldots, b_{n-\delta-1}\}$. For each $b_i \in B$ there is at least one distinct vertex $a_i \in A$ such that $ua_i \rightarrow b_i$.
- Let $b_r$ and $b_s$ be vertices in $B$ such that $b_r b_s \rightarrow u$. Note that $ua_i \rightarrow b_i$ implies that $a_i$, $i \neq r, s$, is adjacent to both $b_r$ and $b_s$. 
Thus, $b_r$ and $b_s$ have at least $n - \delta - 3$ common neighbors in $A$, that is $|N(b_r) \cap N(b_s)| \geq n - \delta - 3$.

By (4) we have

$$n \leq \deg(b_r) + \deg(b_s) - (n - \delta - 3) + 1 \leq 2k - n + \delta + 4,$$

which yields $n \leq k + \frac{1}{2} \delta + 2$. \qed
Since $\delta(G) \leq \Delta(G)$, Lemma 20 can be adapted to provide a better lower bound for $\Delta(G)$ in terms of $|G|$ in the case of Crown graphs.

**Corollary 21**

Let $G$ be a Crown graph. Then $\Delta(G) \geq \left\lceil \frac{2}{3}(|G| - 2) \right\rceil$. 
When equality holds in Corollary 21, then Lemma 20 yields $\delta(G) = \Delta(G)$ so that we have the following interesting result.

**Corollary 22**

Let $G$ be a Crown graph with $\Delta(G) = \lceil \frac{2}{3}(|G| - 2) \rceil$. Then $G$ is regular.

We also establish a lower bound for $\delta(G)$ in terms of $|G|$, namely:

**Lemma 23**

Let $G$ be a Crown graph. Then $\delta(G) \geq \lceil \frac{1}{2}(|G| - 1) \rceil$. 
When, in Lemma 23, equality is attained, the result of Corollary 21 can be further improved as follows.

**Corollary 24**

Let $G$ be a Crown graph with $\delta(G) = \frac{1}{2}(|G| - 1)$. Then

$$\Delta(G) \geq \left\lceil \frac{3|G|-7}{4} \right\rceil.$$

**Proof.** From Lemma 20 we have

$$|G| \leq \Delta(G) + \frac{1}{4}(|G| - 1) + 2$$

which is equivalent to $\Delta(G) \geq \frac{3|G|-7}{4}$.
The upper bound provided by Proposition 9, $|G| - 3$, is sharp within the class of Crown graphs, and for $n$ arbitrarily large.

The lower bound provided in Corollary 21 is also sharp for $n$ arbitrarily large. Indeed, it is sufficient to observe that the circulant graphs $C(n)$ defined previously are Crown graphs with $|G| = n$, $\Delta(G) = \frac{2(n - 2)}{3}$.  

$\alpha(G) = 2$

Crown Graphs

Claw-Free Graphs
A graph $G$ is said to be \textit{claw-free} if it does not contain $K_{1,3}$ as an induced subgraph. As 5-cycle-type graphs are claw-free, in view of Theorem 18 we do not expect claw-freeness to lead to any significant improvement in lower and upper bounds for $\Delta(G)$, in general. However, as long as $\alpha(G) > 2$, the lower bound may be improved as shown in the following result.

**Theorem 25**

\textit{Let $G$ be a $3_t$-critical claw-free graph, with $\alpha(G) > 2$. Then $\Delta(G) \geq \left\lceil \frac{2}{3}(|G| - 2) \right\rceil$.}
For claw-free graphs we have the following theorem, similar to Theorem 18 obtained in the case $\alpha(G) = 2$.

**Theorem 26**

Let $n, k \in \mathbb{N}$ such that $n \geq 8$ and $\left\lceil \frac{2}{3}(n - 2) \right\rceil \leq k \leq n - 3$. Then there exists a $3_t$-critical claw-free graph $G$ with $\alpha(G) > 2$, $|G| = n$, and $\Delta(G) = k$. 
While for the case $\alpha(G) = 2$, as well for Crown graphs and for claw-free graphs, we could establish sharp lower bounds on $\Delta(G)$, we still lack a sharp lower bound for the general case, as we feel that the bound $\lceil \frac{1}{2} |G| \rceil$ provided in Theorem 11 is not sharp, starting, probably, from $n = 12$.

It is also somehow bizarre that in the general case, the additional condition $\alpha(G) = 2$ yields a tighter lower bound (Theorem 15), while, within the class of claw-free graphs, the condition $\alpha(G) > 2$ is the one that provides the tighter lower bound (Theorem 25). This fact may be due, again, to the lack of a sharp lower bound for the general case, which, at this point we really feel should be studied, and, possibly, determined.