Improving Brooks’ theorem

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Outline

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You are a warden in a prison with five large cells. You need to put all the inmates into the cells, but to prevent fighting you cannot put a pair of inmates that have fought before into the same cell. Each inmate in the prison has fought with at most six other inmates and none of the inmates who have fought with six others have fought with each other. Under what conditions can you complete your task?
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- plainly, if there is a group of six inmates who have all fought one another, then you cannot complete your task
- is this simple necessary condition sufficient?
Greedy coloring

- $C := \{c_1, c_2, c_3, \ldots\}$ an infinite set of colors
Greedy coloring

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- $G$ has vertices ordered $v_1, v_2, \ldots, v_n$
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- go through the vertices in order, coloring \( v_i \) with the first color not used on a neighbor of \( v_i \)
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For example, say $C := \{\text{red, green, blue, cyan, \ldots}\}$ and $G$ is the 5-cycle:
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![Diagram of 5-cycle with colors]

$G$ has maximum degree $k$, then $v_i$ has at most $k$ colored neighbors, so greedy coloring uses at most $k + 1$ colors.
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\begin{center}
\begin{tikzpicture}
  \node[shape=circle,draw=red,inner sep=0pt,minimum size=6mm] (a) at (0,0) {}; 
  \node[shape=circle,draw=green,inner sep=0pt,minimum size=6mm] (b) at (1,0) {}; 
  \node[shape=circle,draw=red,inner sep=0pt,minimum size=6mm] (c) at (2,0) {}; 
  \node[shape=circle,draw=green,inner sep=0pt,minimum size=6mm] (d) at (3,0) {}; 
  \node[shape=circle,draw=red,inner sep=0pt,minimum size=6mm] (e) at (4,0) {}; 

  \draw[black, thick] (a) -- (b) -- (c) -- (d) -- (e); 
  \draw[black, thick, bend right=60] (e) to (a); 
\end{tikzpicture}
\end{center}
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- if $G$ has maximum degree $k$, then $v_i$ has at most $k$ colored neighbors, so greedy coloring uses at most $k + 1$ colors
Brooks’ theorem

- \( \chi(G) := \) the minimum number of colors needed to color the vertices of \( G \) so that adjacent vertices receive different colors
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Brooks’ theorem

• \( \chi(G) \) := the minimum number of colors needed to color the vertices of \( G \) so that adjacent vertices receive different colors

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• $\chi(G) \coloneqq$ the minimum number of colors needed to color the vertices of $G$ so that adjacent vertices receive different colors

• $\omega(G) \coloneqq$ the number of vertices in a largest complete subgraph of $G$

• $\Delta(G) \coloneqq$ the maximum degree of $G$
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- \( \omega(G) \) := the number of vertices in a largest complete subgraph of \( G \)
- \( \Delta(G) \) := the maximum degree of \( G \)

<table>
<thead>
<tr>
<th>Theorem (Brooks 1941)</th>
</tr>
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<tbody>
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<td>Every graph with ( \Delta \geq 3 ) satisfies ( \chi \leq \max{\omega, \Delta} ).</td>
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Proof sketch

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The Ore-degree

Definition

The **Ore-degree** of an edge $xy$ in a graph $G$ is

$$\theta(xy) := d(x) + d(y).$$

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- every graph satisfies $\left\lfloor \frac{\theta}{2} \right\rfloor \leq \Delta$
- greedy coloring (in any order) shows that every graph satisfies $\chi \leq \left\lfloor \frac{\theta}{2} \right\rfloor + 1$
Kierstead and Kostochka’s generalization

Theorem (Kierstead and Kostochka 2009)

Every graph with $\theta \geq 12$ satisfies $\chi \leq \max \{\omega, \left\lfloor \frac{\theta}{2} \right\rfloor \}$. 
Kierstead and Kostochka’s generalization

Theorem (Kierstead and Kostochka 2009)

*Every graph with $\theta \geq 12$ satisfies $\chi \leq \max \{\omega, \left\lfloor \frac{\theta}{2} \right\rfloor \}$.*

Kierstead and Kostochka conjectured that the 12 could be reduced to 10. That this would be best possible can be seen from the following example which has $\theta = 9$, $\omega = 4$ and $\chi = 5$. 
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![Graph](image)

Figure: $O_5$, a counterexample with $\theta = 9$. 

Rephrasing the problem

**Definition**

A graph $G$ is called *vertex critical* if $\chi(G - v) < \chi(G)$ for each $v \in V(G)$.
Rephrasing the problem

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Definition

Let $G$ be a vertex critical graph. The low vertex subgraph $L(G)$ is the graph induced on the vertices of degree $\chi(G) - 1$. The high vertex subgraph $H(G)$ is the graph induced on the vertices of degree at least $\chi(G)$.
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## Rephrasing the problem

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Let $G$ be a vertex critical graph. The **low vertex subgraph** $\mathcal{L}(G)$ is the graph induced on the vertices of degree $\chi(G) - 1$. The **high vertex subgraph** $\mathcal{H}(G)$ is the graph induced on the vertices of degree at least $\chi(G)$.

### Problem

Prove that $K_{\Delta(G) + 1}$ is the only vertex critical graph $G$ with $\chi(G) \geq \Delta(G) \geq 6$ such that $\mathcal{H}(G)$ is edgeless.
Kierstead and Kostochka’s proof

- the proof is high-tech and clean, it uses both of the following
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  • Alon and Tarsi’s algebraic list coloring theorem
  • a result of Stiebitz from 1982 proving a conjecture of Gallai stating that $\mathcal{H}(G)$ has at most as many components as $\mathcal{L}(G)$
  • using these it is basically just a counting argument
  • unfortunately, it only works for $\Delta \geq 7$
To get down to $\Delta = 6$, go low-tech and get dirty.
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**Theorem (R. 2010)**

\[
K_{\Delta(G)+1} \text{ is the only vertex critical graph } G \text{ with } \\
\chi(G) \geq \Delta(G) \geq 6 \text{ and } \omega(\mathcal{H}(G)) \leq \left\lfloor \frac{\Delta(G)}{2} \right\rfloor - 2.
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- setting $\omega(H(G)) = 1$ proves Kierstead and Kostochka’s conjecture
To get down to $\Delta = 6$, go low-tech and get dirty.

**Theorem (R. 2010)**

$K_{\Delta(G) + 1}$ is the only vertex critical graph $G$ with $
\chi(G) \geq \Delta(G) \geq 6$ and $\omega(H(G)) \leq \left\lfloor \frac{\Delta(G)}{2} \right\rfloor - 2$.

- setting $\omega(H(G)) = 1$ proves Kierstead and Kostochka’s conjecture
- equivalently, as long as there is no group of six inmates who have all fought one another, you (the warden) can complete your inmate-cell-assignment task
Proof outline

- start with a minimal counterexample $G$
Proof outline

• start with a minimal counterexample $G$
• for any induced subgraph $H$, $\Delta - 1$ coloring $G - H$ leaves a list assignment $L$ on $H$ where $|L(v)| \geq \deg(v) - 1$
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Construct a subgraph $H$ for which such a list assignment can always be completed.
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- first, use minimality of $G$ to exclude some troublesome $H$’s
- run the following recoloring algorithm to construct $H$
Definition

Let $G$ be a vertex critical graph. Let $a \geq 1$ and $r_1, \ldots, r_a$ be such that $1 + \sum_i r_i = \chi(G)$. By a $(r_1, \ldots, r_a)$-partitioned coloring of $G$ we mean a proper coloring of $G$ of the form

$$\{\{x\}, L_{11}, L_{12}, \ldots, L_{1r_1}, L_{21}, L_{22}, \ldots, L_{2r_2}, \ldots, L_{a1}, L_{a2}, \ldots, L_{ar_a}\}.$$

Here $\{x\}$ is a singleton color class and each $L_{ij}$ is a color class.
**Mozhan’s Lemma**

**Lemma (Mozhan 1983)**

Let $G$ be a vertex critical graph. Let $a \geq 1$ and $r_1, \ldots, r_a$ be such that $1 + \sum_i r_i = \chi(G)$. Of all $(r_1, \ldots, r_a)$-partitioned colorings of $G$ pick one minimizing

$$\sum_{i=1}^{a} \left\| G \left[ \bigcup_{j=1}^{r_i} L_{ij} \right] \right\| .$$

Remember that $\{x\}$ is a singleton color class in the coloring. Put $U_i := \bigcup_{j=1}^{r_i} L_{ij}$ and let $Z_i(x)$ be the component of $x$ in $G[\{x\} \cup U_i]$. If $d_{Z_i(x)}(x) = r_i$, then $Z_i(x)$ is complete if $r_i \geq 3$ and $Z_i(x)$ is an odd cycle if $r_i = 2$. 
• take a \(\left(\left\lfloor \frac{\Delta-1}{2} \right\rfloor, \left\lceil \frac{\Delta-1}{2} \right\rceil\right)\)-partitioned coloring minimizing the above function
The recoloring algorithm

- take a \((\lfloor \frac{\Delta-1}{2} \rfloor, \lceil \frac{\Delta-1}{2} \rceil)\)-partitioned coloring minimizing the above function
- prove that we may assume that \(x\) is a low vertex
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- prove that we may assume that \(x\) is a low vertex
- by Mozhan’s lemma, the component of \(x\) in each part induces a clique or an odd cycle
The recoloring algorithm

- swap $x$ with a low vertex $x_1$ in the right part
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- swap $x$ with a low vertex $x_1$ in the right part
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• use the fact that you wrapped around to show that there are many edges between the two cliques
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we have now constructed the desired large "dense" subgraph
Generalizing maximum degree

Definition

For $0 \leq \epsilon \leq 1$, define $\Delta_\epsilon(G)$ as

$$\left\lfloor \max_{xy \in E(G)} (1 - \epsilon) \min\{d(x), d(y)\} + \epsilon \max\{d(x), d(y)\} \right\rfloor.$$
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Note that $\Delta_1 = \Delta$, $\Delta_{\frac{1}{2}} = \left\lfloor \frac{\theta}{2} \right\rfloor$. 
The generalized bound

Theorem (R. 2010)

For every $0 < \epsilon \leq 1$, there exists $t_\epsilon$ such that every graph with
$\Delta_\epsilon \geq t_\epsilon$ satisfies $\chi \leq \max\{\omega, \Delta_\epsilon\}$. 
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**Theorem (R. 2010)**

*For every $0 < \epsilon \leq 1$, there exists $t_\epsilon$ such that every graph with $\Delta_\epsilon \geq t_\epsilon$ satisfies $\chi \leq \max\{\omega, \Delta_\epsilon\}$.***

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- the graph $O_5$ shows that $t_\epsilon = 6$ is smallest for $\frac{1}{2} \leq \epsilon < 1$
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- best known general bounds, $\frac{2}{\epsilon} + 1 \leq t_\epsilon \leq \frac{4}{\epsilon} + 2$
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The lower bound on $t_\epsilon$

Figure: The graph $O_n$. 

$K_{n-2}$
The lower bound on $t_\epsilon$

Figure: The graph $O_n$.

- $\chi(O_n) = n > \omega(O_n)$ and $\Delta(O_n) = \left\lceil \frac{n-1}{2} \right\rceil + n - 2$
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• $\chi(O_n) = n > \omega(O_n)$ and $\Delta(O_n) = \left\lceil \frac{n-1}{2} \right\rceil + n - 2$

• $\mathcal{H}(O_n)$ is edgeless

Figure: The graph $O_n$. 

$K_{n-2}$

$K_{\left\lceil \frac{n-1}{2} \right\rceil}$

$K_{\left\lfloor \frac{n-1}{2} \right\rfloor}$
The lower bound on $t_\epsilon$

- $\chi(O_n) = n > \omega(O_n)$ and $\Delta(O_n) = \left\lceil \frac{n-1}{2} \right\rceil + n - 2$
- $\mathcal{H}(O_n)$ is edgeless
- computing $\Delta_\epsilon$ gives $t_\epsilon \geq \frac{2}{\epsilon} + 1$
What about $\Delta_0$?

- the above proofs only work for $\epsilon > 0$
What about $\Delta_0$?

• the above proofs only work for $\epsilon > 0$
• what happens when $\epsilon = 0$?
What about $\Delta_0$?

- the above proofs only work for $\epsilon > 0$
- what happens when $\epsilon = 0$?
- the parameter $\Delta_0$ has already been investigated by Stacho under the name $\Delta_2$
What about $\Delta_0$?

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**Definition (Stacho 2001)**

For a graph $G$ define

$$\Delta_0(G) := \max_{xy \in E(G)} \min\{d(x), d(y)\}.$$
Facts about $\Delta_0$

- greedy coloring (in any order) shows that every graph satisfies $\chi \leq \Delta_0 + 1$
Facts about $\Delta_0$

- greedy coloring (in any order) shows that every graph satisfies $\chi \leq \Delta_0 + 1$
- for any fixed $t \geq 3$, the problem of determining whether or not $\chi(G) \leq \Delta_0(G)$ for graphs with $\Delta_0(G) = t$ is \textit{NP}-complete (Stacho 2001)
A tempting thought

There exists \( t \) such that every graph with \( \Delta_0 \geq t \) satisfies
\[
\chi \leq \max\{\omega, \Delta_0\}.
\]
A tempting thought

There exists $t$ such that every graph with $\Delta_0 \geq t$ satisfies $\chi \leq \max\{\omega, \Delta_0\}$.

- since $t_\epsilon \geq \frac{2}{\epsilon} + 1$, we see that $t_\epsilon \to \infty$ as $\epsilon \to 0$
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- there is a cute algorithmic way to see this assuming \( P \neq NP \)
- we use Lovász’s \( \vartheta \) parameter which can be approximated in polynomial time and has the property that
\[ \omega(G) \leq \vartheta(G) \leq \chi(G) \]
A polynomial-time algorithm

- assume the tempting thought holds for some \( t \geq 3 \)
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- assume the tempting thought holds for some $t \geq 3$
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A polynomial-time algorithm

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- now, \( \chi \leq \max\{\omega, \Delta_0\} \leq \Delta_0 \)
- we just gave a polynomial time algorithm to determine whether or not \( \chi \leq \Delta_0 \) for graphs with \( \Delta_0 \geq t \)
- this is impossible unless \( P=NP \)
What we can prove about $\Delta_0$

Theorem (R. 2010)

Every graph with $\Delta \geq 3$ satisfies

$$\chi \leq \max \left\{ \omega, \Delta_0, \frac{5}{6}(\Delta + 1) \right\}.$$
What we can prove about $\Delta_0$

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*Every graph with $\Delta \geq 3$ satisfies*

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- the proof uses a recoloring algorithm similar to the above
Improving Brooks’ theorem

Landon Rabern

A prison problem

Some background

The Ore-degree

Rephrasing the problem

Solving the rephrased problem

A spectrum of generalizations

Generalizing maximum degree

The generalized bound

The lower bound on $t_\epsilon$

What about $\Delta_0$?

Further improvements

What we can prove about $\Delta_0$

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Every graph with $\Delta \geq 3$ satisfies

$$\chi \leq \max \left\{ \omega, \Delta_0, \frac{5}{6}(\Delta + 1) \right\}.$$ 

- the proof uses a recoloring algorithm similar to the above
- actually, all the above results about $\Delta_\epsilon$ follow from this result
In joint work with Kostochka and Stiebitz similar techniques were used to improve the bounds further. Highlights:
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Theorem (Kostochka, R. and Stiebitz 2010)

Every graph with $\theta \geq 8$, except $O_5$, satisfies $\chi \leq \max \{\omega, \left\lfloor \frac{\theta}{2} \right\rfloor\}$. 
In joint work with Kostochka and Stiebitz similar techniques were used to improve the bounds further. Highlights:

**Theorem (Kostochka, R. and Stiebitz 2010)**

*Every graph with \( \theta \geq 8 \), except \( O_5 \), satisfies \( \chi \leq \max \{ \omega, \left\lfloor \frac{\theta}{2} \right\rfloor \} \).*

**Theorem (Kostochka, R. and Stiebitz 2010)**

*Every graph satisfies*

\[
\chi \leq \max \left\{ \omega, \Delta_0, \frac{3}{4}(\Delta + 2) \right\}.
\]
Conjecture

Every graph satisfies

\[ \chi \leq \max \left\{ \omega, \Delta_0, \frac{2\Delta + 5}{3} \right\}. \]

The examples \( O_n \) above show that this would be tight.
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Further improvements

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L. Stacho.

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