On a Conjecture of Murty and Simon on Diameter Two Critical Graphs

Teresa W. Haynes

East Tennessee State University

October 5, 2010
Distance parameters in graphs have many real world applications, for example, network design. Perhaps the most studied distance parameter is the diameter of a graph $G$, that is, the length of the longest distance between any pair of vertices of $G$. The diameter of a graph model of a network is extremely important to its overall function as a measure of the worst case number of hops necessary to pass a message between a pair of nodes in the network.
A factor in network design is the behavior of a graph parameter in the presence of a fault. A faulty link in a network can be analyzed by determining the effect of removing an edge from its underlying graph model.
The main theme of this talk is the long-standing conjecture of Murty and Simon concerning the maximum number of edges in a graph for which the removal of an arbitrary edge increases the diameter.
Diameter 2-critical Graphs

A graph $G$ is called *diameter 2-critical* if its diameter is 2, and the deletion of any edge increases the diameter.

Example: Complete Bipartite Graphs
The Murty-Simon Conjecture

(Murty-Simon Conjecture 1979) If $G$ is a diameter 2-critical graph with order $n$ and size $m$, then $m \leq \frac{n^2}{4}$, with equality if and only if $n$ is even and $G$ is the complete bipartite graph $K_{\frac{n}{2}, \frac{n}{2}}$. 
Several attempts have been made to solve the conjecture, and some impressive partial results, supporting the truth of the conjecture, have been obtained.
Brief History of Conjecture

- Erdős told Füredi that this conjecture goes back to the work of Ore in the 1960s.
Brief History of Conjecture

- Erdős told Füredi that this conjecture goes back to the work of Ore in the 1960s.
- 1975: Plesník proved that $m < 3n(n - 1)/8$. 
Brief History of Conjecture

- Erdős told Füredi that this conjecture goes back to the work of Ore in the 1960s.
- 1975: Plesník proved that $m < 3n(n - 1)/8$.
- 1979: Caccetta and Häggkvist showed $m < .27n^2$. 

Teresa W. Haynes
On a Conjecture of Murty and Simon on Diameter Two Critical Graphs
Brief History of Conjecture

- Erdős told Füredi that this conjecture goes back to the work of Ore in the 1960s.
- 1975: Plesník proved that $m < \frac{3n(n-1)}{8}$.
- 1979: Caccetta and Häggkvist showed $m < .27n^2$.
- 1984: Xu published an incorrect proof.
Erdős told Füredi that this conjecture goes back to the work of Ore in the 1960s.

1975: Plesník proved that \( m < \frac{3n(n - 1)}{8} \).

1979: Caccetta and Häggkvist showed \( m < 0.27n^2 \).

1984: Xu published an incorrect proof.

1987: Fan proved the first part of the conjecture for \( n \leq 24 \) and for \( n = 26 \). For \( n \geq 25 \), he obtained
\[ m < \frac{n^2}{4} + \frac{(n^2 - 16.2n + 56)}{320} < 0.2532n^2. \]
Brief History of Conjecture

- Erdős told Füredi that this conjecture goes back to the work of Ore in the 1960s.
- 1975: Plesnícík proved that $m < 3n(n - 1)/8$.
- 1979: Caccetta and Häggkvist showed $m < .27n^2$.
- 1984: Xu published an incorrect proof.
- 1987: Fan proved the first part of the conjecture for $n \leq 24$ and for $n = 26$. For $n \geq 25$, he obtained $m < n^2/4 + (n^2 - 16.2n + 56)/320 < .2532n^2$.
- 1992: Füredi gave an asymptotic result proving the conjecture is true for large $n$, that is, for $n > n_0$ where $n_0$ is a tower of 2's of height about $10^{14}$. 
A Different Perspective

“A little perspective, like a little humor, goes a long way.”
Allen Klein

Attacking the conjecture head-on has not solved it. Perhaps it is time for a new approach.
Total Domination

A *total dominating set* of a graph $G$ with no isolated vertex is a set $S$ of vertices of $G$ such that every vertex of $G$ is adjacent to a vertex in $S$.

The *total domination number* $\gamma_t(G)$ is the minimum cardinality of any total dominating set of $G$.

Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi and is now well studied in graph theory.
Lucas van der Merwe’s Ph.D. Dissertation

A graph $G$ is total domination edge critical if the addition of any edge changes the total domination number.

$k_t$-critical
Decreases By At Most 2

- Clearly, adding an edge cannot increase the total domination number.
Clearly, adding an edge cannot increase the total domination number.

van der Merwe showed that the addition of an edge to a graph can decrease the total domination number by at most two.
Decrees By At Most 2

- Clearly, adding an edge cannot increase the total domination number.
- van der Merwe showed that the addition of an edge to a graph can decrease the total domination number by at most two.
- $k_t$-critical graphs for which the addition of any edge decreases the total domination number by two are called $k_t$-supercritical graphs.
The Connection

“The reverse side also has a reverse side.”
Proverb, Japanese

Seemingly unrelated parameters...

**KEY RELATIONSHIP:**

**Theorem** (Hanson and Wang, 2003) A graph is diameter 2-critical if and only if its complement is $3_t$-critical or $4_t$-supercritical.
Examples

EXAMPLE: The self-complementary 5-cycle is both diameter 2-critical and $3_t$-critical.
Examples

Figure: A diameter 2-critical graph $G$ and its $3_t$-critical complement $\overline{G}$.
Examples

Figure: Edge Removed From $G$. 

$G$:  

$\overline{G}$:
Examples

Figure: A $\gamma_t$-set of $\overline{G}$.
Examples

Figure: Edge Added to $\overline{G}$. 
The Connection

**Theorem.** (Van der Merwe, Mynhardt, and H.) A graph $G$ is $4_t$-supercritical if and only if $G$ is the disjoint union of two complete graphs.

NOTE: The complement of a $4_t$-supercritical graph is a complete bipartite graph. The number of edges is minimized when the partite sets are equal in size, and so the Murty-Simon Conjecture holds for this case and a subset of the complements of $4_t$-supercritical graphs yield the extremal graphs of the conjecture.
Equivalent Conjecture

**Conjecture 2:** If $G$ is a $3_t$-critical graph with order $n$ and size $m$, then $m > n(n - 2)/4$. 
3\textsubscript{t}-critical Graphs

When we gave Lucas the problem for his thesis in 1997, the connection to the Murty-Simon Conjecture was not known.

- Lofty Goal: Characterize the \( k_t \)-critical graphs.
When we gave Lucas the problem for his thesis in 1997, the connection to the Murty-Simon Conjecture was not known.

- **Lofty Goal:** Characterize the $k_t$-critical graphs.
- **Limited Scope:** Characterize the $3_t$-critical graphs.
Now that we know the relationship to the Murty-Simon Conjecture, it is not surprising that van der Merwe was unsuccessful in characterizing the $3_t$-critical graphs. His research did however provide the following nice bounds on the diameter of $3_t$-critical graphs.

**Theorem** (Van der Merwe, Mynhardt, and H.) If $G$ is a $3_t$-critical graph, then $2 \leq diam(G) \leq 3$. 
3ᵉ-critical Graphs

**Theorem** (Hanson-Wang Theorem) If $G$ is a 3ᵉ-critical graph of diameter three and of order $n$ and size $m$, then $m \geq n(n - 2)/4$. 
Diameter 3

We approached the problem from this new perspective: that is, attempting to prove Conjecture 2.

NOTE: In order to prove Conjecture 2, we need to show strict inequality in the Hanson-Wang Theorem. Hence an additional edge was necessary to prove the conjecture in this case.
Diameter 3

Indeed a surprising amount of work was required to find this one additional edge.

**Theorem** (H., Henning, van der Merwe, Yeo, 2009) Conjecture 2 is true for graphs with diameter 3.
Open Portion: Diameter

3t-Critical Graphs

Diameter 2

Diameter 3

Solved

Open
Claw-free

Another piece of the puzzle:

**Theorem** (H., Henning, and Yeo, 2010) Conjecture 2 is true for claw-free graphs.

Idea behind proof: Partition into pseudo-cliques.
Open Portion of Problem: Claw-free

Figure: Open Portion of Problem: Claws
Connectivity

We say that $G$ has connectivity $j$ to mean that $G$ is $j$-connected and $G$ has a cutset of cardinality $j$.

**Theorem** (H., Henning, and Yeo, 2010) Conjecture 2 is true for graphs having connectivity at most 3.

This result is surprising in a sense because intuitively one would think it would be easier to establish a larger edge count for graphs having large connectivity; whereas, here we have proven a lower bound on the number of edges of $3_t$-critical graphs with small connectivity.
Open Portion of Problem: Connectivity

Figure: Open Portion of Problem: Connectivity
Progress Made on the 50 Year Old Murty-Simon Conjecture

Several attempts made to solve the Murty-Simon Conjecture by attacking it head-on, that is, from the viewpoint of diameter 2-critical graphs. Some very impressive partial results obtained.
Progress Made on the 50 Year Old Murty-Simon Conjecture

- Several attempts made to solve the Murty-Simon Conjecture by attacking it head-on, that is, from the viewpoint of diameter 2-critical graphs. Some very impressive partial results obtained.

- Nice observation by Hanson and Wang yields an equivalent conjecture, providing a new way to approach the problem from the perspective of total domination.
Progress Made on the 50 Year Old Murty-Simon Conjecture

- Several attempts made to solve the Murty-Simon Conjecture by attacking it head-on, that is, from the viewpoint of diameter 2-critical graphs. Some very impressive partial results obtained.

- Nice observation by Hanson and Wang yields an equivalent conjecture, providing a new way to approach the problem from the perspective of total domination.

- Using this approach, we have verified the Murty-Simon Conjecture for a number of infinite families of graphs, namely, graphs whose complements have diameter two, are claw-free, or have connectivity at most three.
Status of Problem

- Although we have partial solutions, the Murty-Simon Conjecture is still unsolved.
Status of Problem

- Although we have partial solutions, the Murty-Simon Conjecture is still unsolved.
- The fact that it remains is a testimony to its difficulty and that there is still much work to do.
Status of Problem

- Although we have partial solutions, the Murty-Simon Conjecture is still unsolved.
- The fact that it remains is a testimony to its difficulty and that there is still much work to do.
- On the other hand, our preliminary work is evidence that this new approach is promising as it has shed light on the subject and allowed us to obtain some significant partial results.
Status of Problem

- Although we have partial solutions, the Murty-Simon Conjecture is still unsolved.
- The fact that it remains is a testimony to its difficulty and that there is still much work to do.
- On the other hand, our preliminary work is evidence that this new approach is promising as it has shed light on the subject and allowed us to obtain some significant partial results.
- We believe we have only touched the surface of the problem using this key association between the two parameters, but in doing so have begun to understand it better and to develop proof techniques that can be applied in the future.
Status of Problem

- Although we have partial solutions, the Murty-Simon Conjecture is still unsolved.
- The fact that it remains is a testimony to its difficulty and that there is still much work to do.
- On the other hand, our preliminary work is evidence that this new approach is promising as it has shed light on the subject and allowed us to obtain some significant partial results.
- We believe we have only touched the surface of the problem using this key association between the two parameters, but in doing so have begun to understand it better and to develop proof techniques that can be applied in the future.
- We also want to extend these techniques to investigate possible relationships between other seemingly unrelated parameters and/or criticality issues.