Some New Research Directions in Domination

S. Arumugam

National Centre for Advanced Research in
Discrete Mathematics (n-CARDMATH)
Kalasalingam University, Anand Nagar, Krishnankoil-626 190, India.
and
Conjoint Professor
School of Electrical Engineering and Computer Science
The University of Newcastle
NSW 2308, Australia
In this talk we present several new directions and unsolved problems on domination related concepts.

It is well known that for any graph $G$ of order $n$, we have $\gamma + \overline{\gamma} \leq n + 1$ where $\gamma$ and $\overline{\gamma}$ denote respectively the domination numbers of $G$ and its complement $\overline{G}$. Hence it follows that $\gamma \leq \left\lfloor \frac{n}{2} \right\rfloor$ or $\overline{\gamma} \leq \left\lfloor \frac{n}{2} \right\rfloor$. 

S. Arumugam
Some New Research Directions in Domination
Now, if \( G \) has no isolated vertices, then \( \gamma \leq \frac{n}{2} \). We have the following theorem.

**Theorem**

*If both \( G \) and \( \overline{G} \) have no isolated vertices, then \( \gamma + \overline{\gamma} \leq \left\lfloor \frac{n}{2} \right\rfloor + 2 \).*

Similar improvement for several other parameters have since then been reported [3].
We now look at the corresponding multiplicative version: We have \( \gamma \overline{\gamma} \leq n \). Hence it follows that \( \gamma \leq \lfloor \sqrt{n} \rfloor \) or \( \overline{\gamma} \leq \lfloor \sqrt{n} \rfloor \). Thus at least 50% of the graphs satisfy the inequality \( \gamma \leq \lfloor \sqrt{n} \rfloor \). Find necessary/sufficient conditions for graphs to satisfy this inequality. Then, as in the additive situation, is it possible to put restrictions on \( G \) and \( \overline{G} \) such that for this restricted class the inequality \( \gamma \overline{\gamma} \leq n \) gets substantially improved?
Arunugam et al. [4] have initiated a study of the following parameter which combines the concept of domination and coloring.

Let $G$ be a graph with $\chi(G) = k$. Then there exists a $k$-coloring of $G$ such that at least one color class is a dominating set. The maximum number of color classes that are dominating sets, where the maximum is taken over all $k$-colorings of $G$ is called the dominating-color-number of $G$ and is denoted by $d_\chi(G)$. A few basic results on this parameter are given in [4]. Clearly $1 \leq d_\chi \leq \chi$. There is much scope for further work.
What further can be said about graphs with $d_{\chi} = 1$ and $d_{\chi} = \chi$?

In [4] bounds for $d_{\chi}$, properties of graphs with $d_{\chi} = 1$ and NP-Completeness results are given.
We now consider the edge version of Problem 2.

If $G$ is a graph with chromatic index $\chi'(G) = k$, what is the maximum number of color classes that are maximal matchings (or equivalently edge dominating sets) where the maximum is taken over all $k$-edge colorings of $G$? We denote this number by $d_{\chi'}(G)$. The well known Vizing’s theorem states that $\chi' = \Delta$ or $\Delta + 1$. A graph with $\chi' = \Delta$ is of class 1 and with $\chi' = \Delta + 1$ is of class 2. If $G$ is regular and is of class 1, then in any minimum edge coloring of $G$, every color class is a perfect matching and hence $d_{\chi'} = \chi'$. Also every graph can be embedded as an induced subgraph of a graph $H$ with $d_{\chi'}(H) = 1$.

We are in the process of finalizing of our first paper on this concept. (Collaborators: Jay Bagga and Lowell Beineke).
We observe that the edge version of the parameter behaves quite differently, leading to results which have no vertex analogue.
Let $G$ be a graph with domatic number $d$. What is the maximum number of minimal dominating sets, where the maximum is taken over all domatic partitions of $G$ of order $d'$?

We denote this number by $\Lambda(G)$. Clearly $\Lambda(G) = d - 1$ or $d$.

Work on this parameter is in progress (Collaborators: K. Raja Chandrasekar, T.W. Haynes and Michael Henning).
The following another related question.

Does there exist a domatic partition of order $d(G)$ for $G$ in which at least one dominating set is a $\gamma$-set?
Problem 5

Study the edge version of Problem 4.

In fact for any partition problem of the vertex set or the edge set of a graph into sets satisfying a property $P$, one can investigate the maximum number of maximal/minimal $P$-sets where the maximum is taken over all optimal partitions.
Though domination in graphs has been extensively studied, there is not much progress in the study of domination in directed graphs. Such studies in the directed case have concentrated mainly on the concept of kernels because of its applications to game theory and related areas. Ghoshal et al. [2] have given a survey of results on domination in directed graphs. We propose the following new definition for domination in directed graphs.
Let $D = (V, A)$ be a directed graph. A subset $S$ of $V$ is called a dominating set of $D$ if for every vertex $v$ in $V - S$, there exist two vertices, $v_1, v_2$ (not necessarily distinct) in $S$ such that $(v, v_1)$ and $(v_2, v)$ in $A$. The minimum cardinality of a dominating set of $D$ is called the domination number of $D$ and is denoted by $\gamma(D)$. Thus in the context of a network, a dominating set is a subset of $V$ from which we can send or receive message from all the vertices outside $S$. Moreover this concept of domination in directed graphs includes the concept of domination in graphs as a special case.
If $G$ is a graph and $G^*$ is the directed graph obtained from $G$ by replacing each edge $uv$ by a pair of symmetric arcs $(u, v)$ and $(v, u)$, then $\gamma(G^*) = \gamma(G)$. Hence this concept of domination in directed graphs has vast potential for further research. We have just completed our first paper on this concept, which gives the domination chain for directed graphs.

