Delannoy numbers and a combinatorial proof of the orthogonality of the Jacobi polynomials with natural number parameters

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Delannoy numbers

\[ d_{i,j} = d_{i-1,j} + d_{i,j-1} + d_{i-1,j-1} \]

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They count the number of lattice paths from \((0,0)\) to \((m,n)\) using only steps \((1,0)\), \((0,1)\), and \((1,1)\).

\[ d_{n,n} = \sum_{j=0}^{n} \binom{n}{j} \binom{n+j}{j} \]

(Defined by Henri Delannoy (1895), Sulanke has \(\geq 29\) interpretations.)
A mysterious relation with the Legendre polynomials

Good (1958), Lawden (1952), Moser and Zayachkowski (1963) observed that

\[ d_{n,n} = P_n(3), \]

where \( P_n(x) \) is the \( n \)-th Legendre polynomial.

There has been a consensus that this link is not very relevant.

Banderier and Schwer (2004): “there is no “natural” correspondence between Legendre polynomials and these lattice paths.”

Sulanke (2003): “the definition of Legendre polynomials does not appear to foster any combinatorial interpretation leading to enumeration”.

Jacobi and Legendre polynomials

Usual definition of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$:

$$P_n^{(\alpha,\beta)}(x) = (-2)^{-n}(n!)^{-1}(1-x)^{-\alpha}(1+x)^{-\beta} \frac{d^n}{dx^n} \left((1-x)^{n+\alpha}(1+x)^{n+\beta}\right).$$

$\alpha, \beta > -1$ “for integrability purposes”, $\alpha = \beta = 0$ gives Legendre.

The formula below extends to all $\alpha, \beta \in \mathbb{C}$ (see Szegő (4.21.2)):

$$P_n^{(\alpha,\beta)}(x) = \sum_j \binom{n + \alpha + \beta + j}{j} \binom{n + \alpha}{n - j} \left(\frac{x - 1}{2}\right)^j.$$

Substitute $\alpha = \beta = 0$:

$$P_n^{(0,0)}(x) = \sum_j \binom{n + j}{j} \binom{n}{j} \left(\frac{x - 1}{2}\right)^j$$

is the $n$-th Legendre polynomial.
Properties of Jacobi polynomials

For $\alpha, \beta > -1$ the Jacobi polynomials $P_{n}^{(\alpha,\beta)}(x)$ form an orthogonal basis with respect to the inner product

$$\langle f, g \rangle := \int_{-1}^{1} f(x) \cdot g(x) \cdot (1 - x)^{\alpha}(1 + x)^{\beta} \, dx.$$

“Swapping rule:”

$$(-1)^{n} P_{n}^{(\alpha,\beta)}(-x) = P_{n}^{(\beta,\alpha)}(x),$$
Asymmetric Delannoy numbers

\[ \tilde{d}_{m,n} := \]

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\( \tilde{d}_{m,n} \) is the number of lattice paths from \((0, 0)\) to \((m, n + 1)\) having steps \((x, y) \in \mathbb{N} \times \mathbb{P}\).

(Variant of A049600 in the On-Line Encyclopedia of Integer Sequences.)
Lemma 1  The asymmetric Delannoy numbers satisfy
\[
\tilde{d}_{m,n} = \sum_{j=0}^{n} \binom{n}{j} \binom{m + j}{j}.
\]

Proof: We are enumerating sequences \((0, 0) = (x_0, y_0), (x_1, y_1), \ldots, (x_j, y_j), (x_{j+1}, y_{j+1}) = (m, n + 1),\) where \(0 \leq j \leq n, 0 = x_0 \leq x_1 \leq \cdots \leq x_j \leq x_{j+1} = m,\) and \(0 = y_0 < y_1 < \cdots < y_j < y_{j+1} = n + 1.\) For a given \(j\) there are \(\binom{m+j}{j}\) ways to choose \(0 = x_0 \leq x_1 \leq \cdots \leq x_j \leq x_{j+1} = m\) and \(\binom{n}{j}\) ways to choose \(0 = y_0 < y_1 < \cdots < y_j < y_{j+1} = n + 1.\)

Since
\[
P_n^{(0,\beta)}(x) = \sum_j \binom{n + \beta + j}{j} \binom{n}{j} \left(\frac{x - 1}{2}\right)^j,
\]
we get
\[
\tilde{d}_{n+\beta,n} = P_n^{(0,\beta)}(3) \text{ for } m \geq n
\]
because \(\frac{3 - 1}{2} = 1.\)
Shifted Jacobi and Legendre polynomials

Shifted Legendre polynomials appear even in Abramowitz-Stegun:

\[ \tilde{P}_n(x) := P_n(2x - 1). \]

Shifted Jacobi polynomials seem to be less widely used:

\[ \tilde{P}_n^{(\alpha, \beta)}(x) := P_n^{(\alpha, \beta)}(2x - 1). \]

Well-known:

\[
\tilde{P}_n(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{n+k}{n} x^k
\]

\[
= \sum_{k=0}^{n} (-1)^{n-k} \binom{n+k}{n-k} \binom{2k}{k} x^k.
\]

Generalization for shifted Jacobi polynomials \((\alpha \in \mathbb{N}, \beta \in \mathbb{C})\):

\[
(x-1)^{\alpha} \tilde{P}_n^{(\alpha, \beta)}(x) = \sum_{k=0}^{n+\alpha} (-1)^{n+\alpha-k} x^k \binom{n+\alpha}{k} \binom{n+\beta+k}{n}.
\]

\[ \Rightarrow \tilde{P}_n^{(0, \beta)}(x) = \sum_{k=0}^{n} (-1)^{n-k} x^k \binom{n}{k} \binom{n+\beta+k}{n}. \]
**Weighted Delannoy numbers**

Let $u, v, w$ be commuting variables. We define the *weighted Delannoy numbers* $d_{m,n}^{u,v,w}$ as the total weight of all Delannoy paths from $(0,0)$ to $(m,n)$, where each step $(0,1)$ has weight $u$, each step $(1,0)$ has weight $v$, and each step $(1,1)$ has weight $w$. The weight of a lattice path is the product of the weights of its steps.

Easy to show:

$$d_{n,n}^{u,v,w} = \sum_{k=0}^{n} \binom{2n-k}{k} \binom{2n-2k}{n-k} u^{n-k} v^{n-k} w^{k}.$$  

Since

$$d_{n,n}^{u,v,w} = (-w)^n d_{n,n}^{u,-v/w,-1} = (-w)^n d_{n,n}^{1,-uv/w,-1}$$

we have

$$d_{n,n}^{1,-uv/w,-1} = \sum_{k=0}^{n} \binom{2n-k}{k} \binom{2n-2k}{n-k} (-\frac{uv}{w})^{n-k} (-1)^k.$$  

$$d_{n,n}^{u,v,w} = (-w)^n \tilde{P}_n \left(-\frac{uv}{w}\right).$$

Now

$$d_{n,n} = d_{n,n}^{1,1,1} = (-1)^n \tilde{P}_n(-1) = (-1)^n P_n(-3) = P_n(3)$$

since $(-1)^n P_n(-x) = P_n(x)$. 
Generalization to shifted Jacobi polynomials

\[ d_{m,n}^{u,v,w} = \sum_{k=0}^{n} \binom{m+n-k}{k} \binom{m+n-2k}{n-k} u^{m-k} v^{n-k} w^k. \]

\[ d_{n+\beta,n}^{u,v,w} = u^\beta (-w)^n \tilde{P}_n^{(0,\beta)} \left( -\frac{uv}{w} \right). \]

Here \( \beta \in \mathbb{Z} \) is any integer satisfying \( \beta \geq -n \).

\[ d_{n+\beta,n} = (-1)^n \tilde{P}_n^{(0,\beta)} (-1) = (-1)^n P_n^{(0,\beta)} (-3). \]

Using the “swapping rule”

\[ (-1)^n P_n^{(\alpha,\beta)} (-x) = P_n^{(\beta,\alpha)} (x), \]

we get

\[ d_{n+\beta,n} = P_n^{(\beta,0)} (3). \]

“Swapped” variant of the formula for weighted Delannoy numbers:

\[ d_{n+\beta,n}^{u,v,w} = u^\beta w^n \tilde{P}_n^{(\beta,0)} \left( \frac{uv}{w} + 1 \right). \]
Many arrays, same diagonal

\[ d_{n,n} = d_{n,n}^{r,2/r,-1} \quad \text{for all } r \in \mathbb{R} \setminus \{0\}, \]

and

\[ d_{n,n} = d_{n,n}^{r,1/r,1} \quad \text{for all } r \in \mathbb{R} \setminus \{0\}. \]
Lattice path model for the shifted Legendre and Jacobi polynomials

\[ \tilde{P}_n(x) = d_{n,n}^{1,x,-1} = d_{n,n}^{1,x,-1} \]
\[ \tilde{P}_n^{(0,\beta)}(x) = d_{n+\beta,n}^{1,x,-1} \]

**Fact:** The Jacobi polynomials \( P_n^{(\alpha,\beta)}(x) \) form an orthogonal basis with respect to the inner product

\[ \langle f, g \rangle := \int_{-1}^{1} f(x) \cdot g(x) \cdot (1 - x)^{\alpha}(1 + x)^{\beta} \, dx. \]

**Goal:** to provide a combinatorial, non-inductive proof of this fact for all \( \alpha, \beta \in \mathbb{N} \)

A linear substitution gives the following equivalent form.

The shifted Jacobi polynomials \( \tilde{P}_n^{(\alpha,\beta)}(x) \) form an orthogonal basis with respect to the inner product

\[ \langle f, g \rangle := \int_{0}^{1} f(x) \cdot g(x) \cdot (1 - x)^{\alpha}x^{\beta} \, dx. \]
The case $\alpha = 0$

Assume $m < n$.

\[
(n + m + \beta + 1)! \int_0^1 x^{m+\beta} \tilde{P}_n^{(0,\beta)}(x) \, dx
\]

\[
= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{n + \beta + k}{n} \frac{(n + m + \beta + 1)!}{m + \beta + k + 1}
\]

total weight of all pairs $(L, \sigma)$ where $L$ is a Delannoy path from $(0,0)$ to $(n+\beta,n)$ and $\sigma$ is a bijection

\{r, a_1, \ldots, a_{n+\beta}, b_1, \ldots, b_m\} \to \{1, \ldots, m + n + \beta + 1\},

subject to:

(i) $\sigma(r) < \sigma(a_i)$ holds for all $i$ such that there is an east step in $L$ from $(i-1,y)$ to $(i,y)$ for some $y$;

(ii) $\sigma(r) < \sigma(b_j)$ holds for $j = 1, 2, \ldots, m$.

Diagonal steps contribute a factor of $(-1)$, all others contribute 1.
Cancelling terms

Cancel the diagonal steps with the \(((1, 0), (0, 1))\) sequences, when possible. You will be left with pairs of lattice paths and permutations such that

(a) \(((1, 0), (0, 1))\) is forbidden;

(b) \(\sigma(r) > \sigma(a_i)\) holds for all \(i\) such that there is a northeast east step in \(L\) from \((i - 1, y)\) to \((i, y + 1)\) for some \(y\).

(b) makes \(\sigma(r)\) unique, (a) makes the lattice path depend on the position of the diagonal steps only (\(\sim\) “rook placements”).
Example

\(\alpha = 0, \ n = 6, \ m = 2.\)
Connection to the orthogonality of Laguerre polynomials

We obtained

\[(n + m + \beta + 1)! \cdot \int_0^1 x^{m+\beta} \cdot \tilde{P}_{n,\beta}^{(0)}(x) \, dx\]

\[= \sum_{k=0}^{n} (-1)^k \binom{n + \beta}{k} \binom{n}{k} \cdot k!(n + m + \beta - k)!\]

The right hand side is

\[\int_{0}^{\infty} x^m l_n^{(\beta)}(x)x^\beta e^{-x} \, dx \quad \text{for all } m, n \in \mathbb{N}.\]

Here

\[l_n^{(\beta)}(x) := \sum_{k=0}^{n} (-1)^k \binom{n + \beta}{k} \binom{n}{k} k! x^{n-k}\]

is the \(n\)-th \textit{generalized Laguerre polynomial} associated to the rectangular board \([n + \beta] \times [n]\).
**Rook polynomials**

**Board:** $B \subseteq [n] \times [n]$. $S \subseteq B$ compatible if no two elements of $S$ agree in either coordinate. The *rook polynomial* of $B$ is

$$r_B(x) := \sum_{k=0}^{n} (-1)^k r_k x^{n-k}$$

where $r_k$ is the number of compatible $k$-subsets of $B$. Let $\mathcal{L}$ be the linear functional defined by $\mathcal{L}(x^n) := n!$. Then

$$\mathcal{L}(p(x)) = \int_0^{\infty} e^{-x} p(x) \, dx$$

and the number of permutations $\pi$ of $[n] \times [n]$ such that no $(i, \pi(i))$ belongs to $B$ is $\mathcal{L}(r_B(x))$.

The rook polynomial of $[n] \times [n]$ is the *Laguerre polynomial*

$$l_n(x) := \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 \frac{k!}{k^2} x^{n-k}$$

(1)

$$l_n(x) = (-1)^n n! L_n(x).$$

Laguerre polynomials form an orthogonal basis:

$$\mathcal{L}(l_m(x)l_n(x)) = \delta_{m,n} n!$$
Just for completeness sake

The right hand side is \((m + \beta)!\) times

\[
p(m) := \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n + \beta)_k (n + m + \beta - k)_{n-k}
\]

\[
= (-1)^n \sum_{k=0}^{n} \binom{n}{k} (n + \beta)_k (-m - \beta - 1)_{n-k}.
\]

The number \((-1)^n p(-m)\) is then the number of ways to select a \(k\)-element subset of an \(n\)-element set and injectively color its elements using \(n + \beta\) colors, then color the remaining \(n - k\) elements injectively, using a disjoint set of \(m - \beta - 1\) colors. Thus

\[
(-1)^n p(-m) = \binom{n + m - 1}{n}
\]

\[
p(m) = (-1)^n \binom{n - m - 1}{n}.
\]
The case $\alpha > 0$

$$(x - 1)^\alpha \tilde{P}_n^{(\alpha, \beta)}(x) = \sum_{i=0}^{\alpha} \binom{\alpha}{i} (-1)^i x^{\alpha - i} \tilde{P}_n^{(0, \alpha + \beta - i)}(x)$$

since both sides are the total weight of all Delannoy paths from $(0, 0)$ to $(n + \alpha + \beta, n + \alpha)$ subject to the restriction that none of the first $\alpha$ steps is an east step.

As a consequence

$$\int_0^1 x^m \cdot \tilde{P}_n^{(\alpha, \beta)}(x) \cdot (1 - x)^\alpha x^\beta \ dx$$

$$= (-1)^\alpha \int_0^1 x^{m+\beta} \cdot \sum_{i=0}^{\alpha} \binom{\alpha}{i} (-1)^i x^{\alpha - i} \tilde{P}_n^{(0, \alpha + \beta - i)}(x) \ dx$$

$$= \sum_{i=0}^{\alpha} \binom{\alpha}{i} (-1)^{\alpha + i} \int_0^1 x^{m+(\alpha + \beta - i)} \cdot \tilde{P}_n^{(0, \alpha + \beta - i)}(x) \ dx.$$
\( \tilde{P}_{n}^{(0,\beta)}(x) \) with negative integer \( \beta \)

For \( \beta \in \mathbb{N} \) and \( n \geq \beta \) we have

\[
\tilde{P}_{n}^{(0,-\beta)}(x) = x^{\beta} \tilde{P}_{n-\beta}(x).
\]

Reason:

\[
\tilde{P}_{n}^{(0,\beta)}(x) = x^{n} d_{n+\beta,n}^{1,1,-1/x},
\]

and we may swap the horizontal and vertical axis.

\[
\begin{align*}
\tilde{P}_{0}^{(0,-6)}(x) &= 1 \\
\tilde{P}_{2}^{(0,-6)}(x) &= 3x^2 - 12x + 10 \\
\tilde{P}_{4}^{(0,-6)}(x) &= 5 - 4x \\
\tilde{P}_{6}^{(0,-6)}(x) &= x^6
\end{align*}
\]

transformed Jacobi polynomials \( \hat{P}_{n}^{(\alpha,\beta)}(x) \):

\[
\hat{P}_{n}^{(\alpha,\beta)}(x) := P_{n}^{(\alpha,\beta)}(2x + 1).
\]

\[
\hat{P}_{n}^{(\alpha,\beta)}(x) = \sum_{j=0}^{n} \binom{n + \alpha + \beta + j}{j} \binom{n + \alpha}{n - j} x^{j}.
\]

Claim: For \( \beta \in \mathbb{N} \) and \( 0 \leq n \leq \beta - 1 \) we have

\[
\hat{P}_{n}^{(0,-\beta)}(x) = \hat{P}_{\beta-1-n}^{(0,-\beta)}(x).
\]
A finite orthogonal polynomial sequence

Let $\beta \geq 2$ be any positive integer and let $\mathcal{L}$ be the linear functional defined defined on the vector space

$\{p(x) \in \mathbb{C}[x] : \deg(p) \leq (\beta - 2)/2\}$ by

$$\mathcal{L}(x^k) = k! \cdot (\beta - 2 - k)! \quad \text{for} \ 0 \leq k \leq \beta - 2.$$ 

Then the transformed Jacobi polynomials

$\{\hat{P}_n^{(0, -\beta)}(x) : 0 \leq n \leq (\beta - 2)/2\}$ form an orthogonal basis in the with respect to inner product

$$\langle f, g \rangle := \mathcal{L}(f \cdot g).$$

For odd $\beta$ we may extend $\mathcal{L}$ and the induced inner product to polynomials of degree at most $(\beta - 1)/2$ by making $\mathcal{L}(x^{\beta-1})$ large enough to make the determinant of the $(\beta + 1)/2 \times (\beta + 1)/2$ matrix

$$
\begin{pmatrix}
\mathcal{L}(x^0) & \mathcal{L}(x^1) & \cdots & \mathcal{L}(x^{(\beta-1)/2}) \\
\mathcal{L}(x^1) & \mathcal{L}(x^2) & \cdots & \mathcal{L}(x^{(\beta-1)/2+1}) \\
\vdots & \vdots & \ddots & \vdots \\
\mathcal{L}(x^{(\beta-1)/2}) & \mathcal{L}(x^{(\beta-1)/2+1}) & \cdots & \mathcal{L}(x^{\beta-1})
\end{pmatrix}
$$

positive. The polynomial $\hat{P}_{(\beta-1)/2}^{(0, -\beta)}(x)$ may then be added to the orthogonal basis.
Elements of the proof

For $0 \leq k \leq \beta - 2$ we have:

$$\mathcal{L}(x^k) = (\beta - 1)! B(k + 1, \beta - 1 - k).$$

Here $B(z, w)$ is the beta function

$$B(z, w) = \frac{\Gamma(z) \Gamma(w)}{\Gamma(z + w)}.$$

Thus we have an inner product for polynomials of degree at most $(\beta - 1)/2$. 

$$\langle f, g \rangle = (\beta - 1)! \int_0^1 f \left( \frac{t}{1-t} \right) \cdot g \left( \frac{t}{1-t} \right) \cdot (1-t)^{\beta-2} \, dt$$
Orthogonality:

\[
\sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{m+j}{m} \binom{\beta - 2 - m - j}{n - m - 1} = 0.
\]

Total weight of all \((X, A, B)\) where

(i) \(X \subseteq \{1, 2, \ldots, n\}\);

(ii) \(A = \{a_1, \ldots, a_m\}\) is an \(m\)-element multiset such that each \(a_i\) belongs to \(X \cup \{0\}\);

(iii) \(B = \{b_1, \ldots, b_{n-m-1}\}\) is an \((n - m - 1)\)-element multiset such that each \(b_j\) belongs to \(\{1, \ldots, \beta - n\} \setminus X\).

The weight of \((X, A, B)\) is \((-1)^{|X|}\). Since \(|A| + |B| = n - 1\), there is \(c \in \{1, \ldots, n\}\) that does not appear in \(A\), nor in \(B\). For each \(X \subset \{1, \ldots, n\} \setminus \{c\}\), the weight of \((X, A, B)\) and of \((X \cup \{c\}, A, B)\) cancel.

Extending to degree \((\beta - 1)/2\) for odd \(\beta\):

Only need to make sure entire matrix has positive determinant, all other principal minors have. The determinant is a linear function of \(\mathcal{L}(x^\beta)\) whose coefficient is positive.
Weighted Schröder numbers

Schröder path from \((0, 0)\) to \((n, n)\): a Delannoy path not going above the line \(y = x\).

**weighted Schröder numbers** \(s_{n}^{u,v,w}\): the total weight of all Schröder paths from \((0, 0)\) to \((n, n)\), where each east step \((0, 1)\) has weight \(u\), each north step has weight \(v\), and each northeast step has weight \(w\).

**Schröder polynomials:** \(S_n(x) := s_{n}^{1,x,-1}\).

\[
S_n(x) = \sum_{j=0}^{n} \frac{(-1)^{n-j}}{j+1} \binom{2j}{j} \frac{n+j}{n-j} x^j \quad \text{for } n \geq 1.
\]

\[
S_n(x) = \frac{1}{n+1} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j+1} \binom{n+j}{n} x^j
\]

For \(n \geq 1\) we also have

\[
(x-1)\tilde{P}_n^{(1,-1)}(x) = \sum_{k=0}^{n+1} (-1)^{n+1-k} x^k \binom{n+1}{k} \binom{n-1+k}{n}
\]

Therefore,

\[
S_n(x) = \frac{x - 1}{(n + 1)x} \tilde{P}_n^{(1,-1)}(x).
\]
Facts about $s_{n}^{u,v,w}$ and $S_{n}(x)$

\[ s_{n}^{u,v,w} = (-w)^{n} S_{n} \left(-\frac{uv}{w}\right) \]

\[ s_{n}^{u,v,w} = \frac{(-w)^{n}}{n + 1} \left(1 + \frac{w}{uv}\right) \tilde{P}_{n}^{(1,-1)} \left(-\frac{uv}{w}\right). \]

\[ s_{n} := s_{n}^{1,1,1} = \frac{(-1)^{n}2}{n + 1} \tilde{P}_{n}^{(1,-1)}(-1) = \frac{(-1)^{n}2}{n + 1} P_{n}^{(1,-1)}(-3) \]

The “swapping rule” yields

\[ s_{n} = \frac{2}{n + 1} \cdot P_{n}^{(-1,1)}(3) \quad \text{for } n \geq 1. \]

\[ d_{n,n}^{u,v,w} = 2uv \sum_{k=0}^{n-1} d_{k,k}^{u,v,w} s_{n-k-1}^{u,v,w} + wd_{n-1,n-1}^{u,v,w}. \]

\[ \tilde{P}_{n}(x) = 2x \sum_{k=0}^{n-1} \tilde{P}_{k}(x) S_{n-k-1}(x) - \tilde{P}_{n-1}(x). \]

\[ \tilde{P}_{n}(x) = 2 \sum_{k=0}^{n-2} \tilde{P}_{k}(x) \frac{x - 1}{n - k} \tilde{P}_{n-k-1}^{(1,-1)}(x) + (2x - 1) \tilde{P}_{n-1}(x) \quad \text{and} \]

\[ P_{n}(x) = \sum_{k=0}^{n-2} P_{k}(x) \frac{x - 1}{n - k} P_{n-k-1}^{(1,-1)}(x) + x P_{n-1}(x) \quad \text{for } n \geq 1. \]
A formula for repeated antiderivatives of the shifted Legendre polynomials

\[ S_n(x) = \frac{1}{x} \int_0^x \tilde{P}_{n-1}(t) \, dt \quad \text{holds for } n \geq 1. \]

Let \( n \) and \( \alpha \) be positive integers. Applying the antiderivative operator

\[ f(x) \mapsto \int_0^x f(t) \, dt \]

to \( \tilde{P}_n(x) \) exactly \( \alpha \) times yields the polynomial

\[ \frac{1}{(n+\alpha)_{\alpha}} (x - 1)^\alpha \tilde{P}_n^{(\alpha,-\alpha)}(x). \]

This follows from

\[ \frac{d}{dx} \frac{(x - 1)^\alpha \tilde{P}_n^{(\alpha,-\alpha)}(x)}{(n + \alpha)_{\alpha}} = \frac{(x - 1)^{\alpha - 1} \tilde{P}_n^{(\alpha - 1,-(\alpha-1))}(x)}{(n + \alpha - 1)_{\alpha-1}} \]

for \( \alpha \geq 1. \)
Favard’s theorem

Favard’s theorem states that a sequence of monic polynomials \( \{p_n(x)\}_{n \geq 0} \) is an orthogonal polynomial sequence, if and only if it satisfies

\[
p_n(x) = (x - c_n)p_{n-1}(x) - \lambda_n p_{n-2}(x) \quad n = 1, 2, 3, \ldots
\]

where \( p_{-1}(x) = 0, p_0(x) = 1 \), the numbers \( c_n \) and \( \lambda_n \) are constants, \( \lambda_n \neq 0 \) for \( n \geq 2 \), and \( \lambda_1 \) is arbitrary. The original proof provides only a recursive description of \( \mathcal{L} \). Viennnot gave a combinatorial proof of Favard’s theorem, upon which he has built a general combinatorial theory of orthogonal polynomials. In his theory, the values \( \mathcal{L}(x^n) \) are explicitly given as sums of weighted Motzkin paths.
Two notes of Favard’s theorem and Viennot’s model

The polynomials \( \{S_n(x)\}_{n \geq 0} \) almost form an orthogonal polynomial sequence.

\[
p_n(x) := \frac{1}{\binom{2n}{n}} \frac{x - 1}{x} \tilde{P}^{(1,-1)}_n(x)
\]

\[
p_n(x) = \left( x - \frac{1}{2} \right) p_{n-1}(x) - \frac{n(n-2)}{4(2n-1)(2n-3)} p_{n-2}(x)
\]

for \( n \geq 2 \). Substituting \( n = 2 \) yields \( \lambda_2 = 0 \).

The monic variant of the Legendre polynomials is

\[
p_n(x) := \frac{2^n P_n(x)}{\binom{2n}{n}}.
\]

Favard’s recursion formula takes the form

\[
p_n(x) = xp_{n-1}x - \frac{(n-1)^2}{(2n-1)(2n-3)} p_{n-2}(x).
\]

Challenge: Consider weighted Motzkin paths from \((0,0)\) to \((n,0)\). The horizontal steps have zero weight, the northeast steps \((1,1)\) have weight 1, the southeast steps \((1,-1)\) have weight \( k^2/(4k^2 - 1) \) if they start at a point whose second coordinate \( k \). Using Viennot’s model, the total weight if these paths should be \( 1/(n+1) \) for all even \( n \in \mathbb{N} \).
Connection to Riordan arrays

A Riordan array is a pair \((d(t), h(t))\) of formal power series in the variable \(t\). These functions define the triangle 
\[d_{n,k} = [t^n]d(t)(th(t))^k.\]

The weighted Delannoy number \(d_{m,n}^{u,v,w}\) is the coefficient of \(t^n\) in \((u + wt)^m/(1 - vt)^{m+1}\). An immediate consequence of this observation is that the \(n\)-th row \(k\)-th column entry in the Riordan array \((1/(1 - vt),\ t(u + wt)/(1 - vt))\) is \(d_{k,n-k}^{u,v,w}\). The numbers \(d_{m,n}^{1,2,-1}\) appear as entry A1016195 in Sloane [16], listing the entries of the Riordan array \((1/(1 - 2t),\ t(1 - t)/(1 - 2t))\). Our results should allow to write summation formulas for Jacobi polynomials using the theory of Riordan arrays.
References


