SOME RECENT RESULTS ON RESTRAINED DOMINATION

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SOME RECENT RESULTS ON RESTRAINED DOMINATION

INTRO TO DOMINATION CONCEPTS

A MORE GENERAL APPROACH

UPPER BOUNDS

LOWER BOUNDS

γ-EXCELLENCE AND RESTRAINED DOMINATION IN TREES

MOTIVATION FOR NORDHAUS-GADDUM RESULTS

PARTITIONS

OTHER RECENT PAPERS

A Nordhaus-Gaddum Result for Total Restrained Domination

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OUTLINE

1. Intro to Domination Concepts
2. A more general approach
3. Upper bounds
4. Lower bounds
5. γ-excellence and restrained domination in trees
6. Motivation for Nordhaus-Gaddum results
   • A Nordhaus-Gaddum Result for Total Restrained Domination
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A Happy Little Tree

**Figure:** A Happy Little Tree
Domination in graphs

**Definition**

- A dominating set $D$ of a graph $G = (V, E)$ is a set of vertices of $V$ for which each $v \in V$ is either in $D$ or adjacent to a vertex of $D$. That is to say, $N[D] = V$.
- The domination number of $G$, denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of $G$.

The concept of domination was first introduced by Berge and Ore.


**Domination in Graphs**

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A DOMINATING SET

FIGURE: D = Shaded vertices

FIGURE: Induced Subgraphs
Total Domination in Graphs

Definition

- A total dominating set $Q$ of a graph $G = (V, E)$ is a set of vertices of $V$ for which each $v \in V$ is adjacent to a vertex of $Q$. That is to say, $N(Q) = V$.
- The total domination number of $G$, denoted by $\gamma_t(G)$, is the smallest cardinality of a total dominating set of $G$.

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Total Domination in Graphs

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A Total Dominating Set

Some Recent Results on Restrainted Domination

Motivation for Nordhaus-Gaddum

Figure: Q = Shaded vertices

Figure: Induced Subgraphs
Restrained Domination in Graphs

Definition

- A restrained dominating set \( R \) of a graph \( G = (V, E) \) is a dominating set for which each \( v \in V - R \) is adjacent to another vertex of \( V - R \).
- The restrained domination number of \( G \), denoted by \( \gamma_r(G) \), is the smallest cardinality of a restrained dominating set of \( G \).

The concept of restrained domination was introduced by Telle, as a vertex-partitioning problem.
**DEFINITION**

- A **restrained dominating set** $R$ of a graph $G = (V, E)$ is a dominating set for which each $v \in V - R$ is adjacent to another vertex of $V - R$.

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A Restrainted Dominating Set

Some recent results on restrained domination

Hattingh

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A more general approach

Upper bounds

Lower bounds

γ-excellence and restrained domination in trees

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Figure: R = Shaded vertices

Figure: Induced Subgraphs
One possible application of the concept of restrained domination is that of prisoners and guards. Here, each vertex not in the restrained dominating set corresponds to a position of a prisoner, and every vertex in the restrained dominating set corresponds to a position of a guard. Note that each prisoner’s position is observed by a guard’s position (to effect security) while each prisoner’s position is seen by at least one other prisoner’s position (to protect the rights of prisoners). To be cost effective, it is desirable to place as few guards as possible (in the sense above).
DEFINITION

- A **total restrained dominating set (TRDS)** $S$ of a graph $G = (V, E)$ is a set for which every vertex $v \in V$ is adjacent to a vertex in $S$ and every vertex of $V - S$ is adjacent to a vertex in $V - S$.

- The **total restrained domination** number of $G$, denoted by $\gamma_{tr}(G)$, is the smallest cardinality of a total restrained dominating set of $G$. 
The concept of total restrained domination was introduced by Chen, Ma and Sun, and has been further studied by Zelinka. We may also note that the concept of total restrained domination was also introduced by Telle and Proskurowski, albeit indirectly, as a vertex partitioning problem.
A TOTAL RESTRAINED DOMINATING SET

**Figure:**
S = Shaded vertices

**Figure:** Induced Subgraphs

### Motivation for Nordhaus-Gaddum

**Some Recent Results on Restrainted Domination**

**Intro to Domination Concepts**

**A More General Approach**

**Upper Bounds**

**Lower Bounds**

γ-Excellence and Restrainted Domination in Trees
Consider an arbitrary set $S$ and its complement $V - S$ in a graph $G = (V, E)$.

Consider also the subgraphs $\langle S \rangle$ and $\langle V - S \rangle$ and the bipartite subgraph $\langle S, V - S \rangle$ generated by all edges between $S$ and $V - S$.

Note that $G = \langle S \rangle \cup \langle V - S \rangle \cup \langle S, V - S \rangle$. 
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Motivation for Nordhaus-Gaddum
A more general approach

- Consider an arbitrary set $S$ and its complement $V - S$ in a graph $G = (V, E)$.
- Consider also the subgraphs $\langle S \rangle$ and $\langle V - S \rangle$ and the bipartite subgraph $\langle S, V - S \rangle$ generated by all edges between $S$ and $V - S$.
- Note that $G = \langle S \rangle \cup \langle V - S \rangle \cup \langle S, V - S \rangle$. 
Consider four values:

- the minimum degree of vertices $v \in \langle S \rangle$, denoted by $\delta_S$,
- the minimum degree of vertices $v \in S$ in $\langle S, V - S \rangle$, denoted by $\delta_{S,V-S}$,
- the minimum degree of vertices $v \in V - S$ in $\langle V - S, S \rangle$, denoted by $\delta_{V-S,S}$, and
- the minimum degree of vertices $v \in \langle V - S \rangle$, denoted by $\delta_{V-S}$.
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- the minimum degree of vertices $v \in \langle V - S \rangle$, denoted by $\delta_{V-S}$. 
A more general approach

Many dominating concepts can be defined in terms of various combinations of these four values. In what follows, X will denote that the particular value does not matter.

<table>
<thead>
<tr>
<th>δ_s</th>
<th>δ_s, v−s</th>
<th>δ_v−s, s</th>
<th>δ_v−s</th>
<th>Type of domination</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>X</td>
<td>≥ 1</td>
<td>X</td>
<td>S is a DS</td>
</tr>
<tr>
<td>= 0</td>
<td>X</td>
<td>≥ 1</td>
<td>X</td>
<td>S is an independent DS</td>
</tr>
<tr>
<td>≥ 1</td>
<td>X</td>
<td>≥ 1</td>
<td>X</td>
<td>S is a total DS</td>
</tr>
<tr>
<td>= 1</td>
<td>X</td>
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<td>X</td>
<td>S is a paired DS</td>
</tr>
<tr>
<td>X</td>
<td>= 1</td>
<td>= 1</td>
<td>X</td>
<td>there is a perfect matching between S and V − S</td>
</tr>
<tr>
<td>X</td>
<td>X</td>
<td>= 1</td>
<td>X</td>
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</tr>
<tr>
<td>X</td>
<td>X</td>
<td>≥ 1</td>
<td>≥ 1</td>
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</tr>
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<tr>
<td>X</td>
<td>X</td>
<td>≤ 1</td>
<td>X</td>
<td>S is a nearly perfect DS</td>
</tr>
<tr>
<td>X</td>
<td>≥ 1</td>
<td>≥ 1</td>
<td>X</td>
<td>both S and V − S are DS</td>
</tr>
<tr>
<td>X</td>
<td>X</td>
<td>≥ n</td>
<td>X</td>
<td>S is an n-DS</td>
</tr>
<tr>
<td>X</td>
<td>≤ k</td>
<td>≥ 1</td>
<td>X</td>
<td>S is a capacity k DS</td>
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</tr>
<tr>
<td>X</td>
<td>$\geq n$</td>
<td>$\geq 1$</td>
<td>X</td>
<td>$S$ is an $n$-DS</td>
</tr>
<tr>
<td>X</td>
<td>$\leq k$</td>
<td>$\geq 1$</td>
<td>X</td>
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\[
\begin{array}{|c|c|c|c|c|}
\hline
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\hline
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= 1 & X & \geq 1 & X & S \text{ is a paired DS} \\
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X & X & = 1 & X & S \text{ is a perfect DS} \\
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\geq 1 & X & \geq 1 & \geq 1 & S \text{ is a total restrained DS} \\
X & X & \leq 1 & X & S \text{ is a nearly perfect DS} \\
X & \geq 1 & \geq 1 & X & \text{both } S \text{ and } V - S \text{ are DS} \\
X & X & \geq n & X & S \text{ is an } n\text{-DS} \\
X & \leq k & \geq 1 & X & S \text{ is a capacity } k \text{ DS} \\
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\end{array}
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A more general approach

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<td>X</td>
<td>S is an independent DS</td>
</tr>
<tr>
<td>$\geq 1$</td>
<td>X</td>
<td>$\geq 1$</td>
<td>X</td>
<td>S is a total DS</td>
</tr>
<tr>
<td>= 1</td>
<td>X</td>
<td>$\geq 1$</td>
<td>X</td>
<td>S is a paired DS</td>
</tr>
<tr>
<td>X</td>
<td>= 1</td>
<td>= 1</td>
<td>X</td>
<td>there is a perfect matching between $S$ and $V - S$</td>
</tr>
<tr>
<td>X</td>
<td>X</td>
<td>= 1</td>
<td>X</td>
<td>S is a perfect DS</td>
</tr>
<tr>
<td>X</td>
<td>X</td>
<td>$\geq 1$</td>
<td>$\geq 1$</td>
<td>S is a restrained DS</td>
</tr>
<tr>
<td>$\geq 1$</td>
<td>X</td>
<td>$\geq 1$</td>
<td>$\geq 1$</td>
<td>S is a total restrained DS</td>
</tr>
<tr>
<td>X</td>
<td>X</td>
<td>$\leq 1$</td>
<td>X</td>
<td>S is a nearly perfect DS</td>
</tr>
<tr>
<td>X</td>
<td>$\geq 1$</td>
<td>$\geq 1$</td>
<td>X</td>
<td>both $S$ and $V - S$ are DS</td>
</tr>
<tr>
<td>X</td>
<td>X</td>
<td>$\geq n$</td>
<td>X</td>
<td>S is an $n$-DS</td>
</tr>
<tr>
<td>X</td>
<td>$\leq k$</td>
<td>$\geq 1$</td>
<td>X</td>
<td>S is a capacity $k$ DS</td>
</tr>
</tbody>
</table>
**Upper Bounds when \( \delta \geq 1 \)**

**Observation**

(Domke, Hedetniemi, Laskar and Markus, 1999)

\[ \gamma_r(G) \leq n \] for any graph \( G \) of order \( n \).

**Proposition**

(Domke, Hedetniemi, Laskar and Markus, 1999) Let \( G \) be a connected graph of order \( n \). Then \( \gamma_r(G) = n \) if and only if \( G \) is a star.
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Upper bounds when $\delta \geq 1$

If $G$ is a connected graph of order $n$ and $G$ is not a star, then $\gamma_r(G) \leq n - 2$.

**Definition**

A leaf in a graph is a vertex of degree one, while a stem is a vertex adjacent to a leaf.

**Theorem**

(Domke, Hedetniemi, Laskar and Markus, 1999) If $T$ is a tree of order $n \geq 3$, then $\gamma_r(T) = n - 2$ if and only if $T$ is obtained from $P_4$, $P_5$ or $P_6$ by adding zero or more leaves to the stems of the path.
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**UPPER BOUNDS WHEN \( \delta \geq 1 \)**

If \( G \) is a connected graph of order \( n \) and \( G \) is not a star, then 
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**THEOREM**

(Domke, Hattingh, Henning and Markus, 2000) Let $G$ be a connected graph of order $n$ with $\delta \geq 2$. If $G \notin \{B_1, \ldots, B_8\}$, then $\gamma_r(G) \leq \frac{n-1}{2}$. 

**UPPER BOUNDS WHEN $\delta \geq 2$**

Some recent results on restrained domination concepts and upper bounds.

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**Upper bounds**

**Lower bounds**

$\gamma$-excellence and restrained domination in trees.

Motivation for Nordhaus-Gaddum.
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**Upper Bounds when $\delta \geq 2$**
**Upper Bounds when $\delta \geq 2$**

**Theorem**

(Cockayne, unpublished) *If $G$ has order $n$ and minimum degree $\delta$, then*

$$
\gamma_r(G) \leq \begin{cases} 
  n \left(1 - \frac{2\delta}{(2\delta+2)^{1+\frac{1}{\delta}}} \right) & \text{if } \delta \geq 4 \\
  .895n & \text{if } \delta = 2 \\
  .668n & \text{if } \delta = 3 
\end{cases}
$$

Let $\mathcal{K}$ be the set of all even order complete graphs of order at least six with a one factor removed.

**Theorem**

(Hattingh and Joubert, 2008) *Let $G$ be a connected graph of order $n$ with $\delta \geq 2$. If $G \notin \{B_1, \ldots, B_8\} \cup \mathcal{K}$, then*

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Upper bounds when $\delta \geq 2$

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A graph $G$ is said to be **claw-free** if for any vertex $u$ of degree at least three we have that if $v, w, x \in N(u)$ then $\langle \{v, w, x, u\} \rangle$ is not isomorphic to $K_{1,3}$. 

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**Upper Bounds**

**Lower Bounds**

$\gamma$-Excellence and Restrained Domination in Trees

**Motivation for Nordhaus-Gaddum**
THEOREM

(Hattingh and Joubert, 2008) Let $G$ be a connected claw-free graph of order $n$ with $\delta \geq 2$. If $G \notin \{C_4, C_7\} \cup \{C_5, C_8, \ldots, C_{17}\}$, then $\gamma_r(G) \leq \frac{2n}{5}$. Moreover, this bound is best possible.
Theorem

Dankelmann, Day, Hattingh, Henning, Markus and Swart, 2007 If \( G \) is a graph of order \( n \) and minimum degree \( \delta \geq 2 \), then \( \gamma_r(G) \leq n - \Delta \).

Moreover, in the same paper,

- the connected triangle-free graphs of order \( n \) with minimum degree \( \delta \geq 2 \) for which \( \gamma_r(G) = n - \Delta \) are characterized; and
- for connected graphs \( G \) that are both triangle-free and \( C_5 \)-free, a particularly simple characterization is obtained.
An upper bound for restrained domination in terms of order and maximum degree

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Theorem

(Hattingh, Joubert, Loizeaux, Plummer and Van der Merwe, 2008) Let G be a connected graph of order n and size m. Then $\gamma_r(G) \geq n - \frac{2m}{3}$.

Corollary

(Domke, Hattingh, Henning and Markus, 2000) Let T be a tree of order n. Then $\gamma_r(T) \geq \lceil \frac{n+2}{3} \rceil$.

Hattingh and Plummer (2008) constructively characterized trees T of order n for which $\gamma_r(T) = \lceil \frac{n+2}{3} \rceil$. 
Some recent results on restrained domination

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**Lower bounds**
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Hattingh and Plummer (2008) constructively characterized trees $T$ of order $n$ for which $\gamma_r(T) = \lceil \frac{n+2}{3} \rceil$. 
A graph $G$ is called $\gamma$-excellent if every vertex of $G$ belongs to some dominating set of minimum cardinality.
**Definition**

A subset $S \subseteq V$ is a packing in $G$ if the vertices of $S$ are pairwise at distance at least three apart in $G$. The packing number $\rho(G)$ is the maximum cardinality of a packing in $G$. 

![Graph example](image-url)
Some recent results on restrained domination

**Observation**

For a graph $G$, $\gamma(G) \leq \gamma_r(G)$.

**Definition**

A graph $G$ such that $\gamma(G) = \gamma_r(G)$ is called a $(\gamma, \gamma_r)$-graph.
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A graph $G$ such that $\gamma(G) = \gamma_r(G)$ is called a $(\gamma, \gamma_r)$-graph.
**Tree labelings and a first operation**

**Definition**

A labeling of a tree $T$ is a function $S: V(T) \rightarrow \{A, B\}$. The label of a vertex $v$ is also called its status, denoted $sta(v)$. By a labeled $K_1$ we shall mean a $K_1$ whose vertex is labeled with status $B$.

**Operation $O_1$.** Attach to a vertex $v$ of status $A$ a path $v, x, y$ where $sta(x) = A$ and $sta(y) = B$. 

$$O_1: \quad \begin{array}{c}
\bullet \\
\circ \\
\bullet \\
\end{array}$$
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\begin{array}{c}
A \\
\end{array} \\
\begin{array}{c}
A \\
\end{array} \\
\begin{array}{c}
B \\
\end{array}
\end{array}$$
**SECOND OPERATION AND A FAMILY OF TREES**

**Operation** $O_2$. Attach to a vertex $v$ of status $B$ a path $v, x, y, z$ where $\text{sta}(x) = \text{sta}(y) = A$ and $\text{sta}(z) = B$.

![Diagram](image)

Let $T$ be the family of trees that can be labeled so that the resulting family of labeled trees contains a labeled $K_1$ and is closed under the two operations $O_1$ and $O_2$ listed above, which extend the tree $T$ by attaching a tree to the vertex $v \in V(T)$, called the **attacher**.
**Second Operation and a Family of Trees**

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Let $\mathcal{T}$ be the family of trees that can be labeled so that the resulting family of labeled trees contains a labeled $K_1$ and is closed under the two operations $O_1$ and $O_2$ listed above, which extend the tree $T$ by attaching a tree to the vertex $v \in V(T)$, called the **attacher**.
**A MORE GENERAL APPROACH**

**UPPER BOUNDS**

**LOWER BOUNDS**

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**MOTIVATION FOR NORDHAUS-GADDUM**

**SOME RECENT RESULTS ON RESTRAINED DOMINATION**

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**AN EXAMPLE OF A GRAPH IN $\mathcal{T}$**

**Figure:** The base graph $K_1$ labeled $B$
**AN EXAMPLE OF A GRAPH IN \( \mathcal{T} \)**

**FIGURE:** Applying Operation \( O_2 \)
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Figure: Applying Operation $O_1$
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Figure: Applying Operation \( O_1 \)
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Figure:
**RESULT**

**THEOREM**

*(Dankelmann, Hattingh, Henning, and Swart, 2007)* Let $T$ be a tree. Then the following statements are equivalent:

1. $T \in \mathcal{T}$;
2. $T$ has a unique $\rho(T)$-set and this set is a dominating set of $T$;
3. $T$ is a $(\gamma, \gamma_r)$-tree;
4. $T$ is $\gamma$-excellent and $T \neq K_2$. 

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**Result**
In 1956 Nordhaus and Gaddum presented best possible bounds on the sum of the chromatic number of a graph and its complement. The corresponding result for the domination number was presented by Jaeger and Payan in 1972.
Nordhaus-Gaddum Type Example

Figure: $K_4$
Nordhaus-Gaddum Type Example

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Figure: $K_4$
Thus, we have the red graph $K$ and the blue graph $\bar{K}$.

**Figure:** $K$

**Figure:** $\bar{K}$
Notice that $\gamma(K) = 2$. 

**Figure: $K$**

**Figure: $\overline{K}$**
Some recent results on restrained domination.

Motivation for Nordhaus-Gaddum type examples.

Also, notice that $\gamma(\bar{K}) = 1$.

**Figure: $K$**

**Figure: $\bar{K}$**
Hence, $\gamma(K) + \gamma(\overline{K}) = 2 + 1 = 3$. 

**Figure:** $K$ 

**Figure:** $\overline{K}$
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1. A Nordhaus-Gaddum Result for Total Restrained Domination
2. A Nordhaus-Gaddum Result for Restrained Domination

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OTHER RECENT PAPERS
A best possible bound on the sum of the total restrained domination numbers of a graph and its complement was obtained by Chen, Ma and Sun.

**Theorem**

If $G$ is a graph of order $n \geq 2$ such that neither $G$ nor $\overline{G}$ contains isolated vertices or has diameter two, then

$$\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n + 4.$$

Yet, this theorem is incorrect.
Motivation

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COUNTEREXAMPLE

**Figure:** $K_5$
**SOKE RECENT RESULTS ON RESTRAINED DOMINATION**

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**MOTIVATION FOR NORDHAUS-GADDUM**

**COUNTEREXAMPLE**

**Figure: $K_5$**
Notice that $K$ is self-complementary.

**Figure:** Red $K$

**Figure:** Blue $K$
All vertices are required for the red $K$.

**Figure**: Red $K$

**Figure**: Blue $K$
Similarly, all vertices are required for the blue $K$.

**Figure:** Red $K$

**Figure:** Blue $K$
COUNTEREXAMPLE

**Observation**

- Notice that $\gamma_{tr}(K) = \gamma_{tr}(\bar{K}) = 5$.
- Hence, $\gamma_{tr}(K) + \gamma_{tr}(\bar{K}) = 5 + 5 > n(K) + 4$.
- Since $K$ has neither an isolated vertex nor diameter two, there is a problem.

We amend this theorem to account for $K$, and moreover include graphs of diameter two.
COUNTEREXAMPLE

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Amended Theorem

(Hattingh, Jonck, Joubert and Plummer, 2008) Let $G$ be a graph of order $n$ such that neither $G$ nor $\overline{G}$ contains isolated vertices or is isomorphic to $K$. Then

$$4 \leq \gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n + 4.$$

Our proof is predicated on the level decomposition of a graph.
Amended Theorem

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Our proof is predicated on the level decomposition of a graph.
Furthermore, we characterize the extremal graphs.

**Characterization**

- Let $n \geq 5$ be an integer and suppose $\{x, y, u, v\}$ and $X$ are disjoint sets of vertices such that $|X| = n - 4$.
- Let $\mathcal{L}$ be the family of graphs $G$ of order $n$ where $V(G) = \{x, y, u, v\} \cup X$ and with the following properties:
Furthermore, we characterize the extremal graphs.

**Characterization**

- Let $n \geq 5$ be an integer and suppose $\{x, y, u, v\}$ and $X$ are disjoint sets of vertices such that $|X| = n - 4$.
- Let $\mathcal{L}$ be the family of graphs $G$ of order $n$ where $V(G) = \{x, y, u, v\} \cup X$ and with the following properties:
Furthermore, we characterize the extremal graphs.

**Characterization**

- Let $n \geq 5$ be an integer and suppose $\{x, y, u, v\}$ and $X$ are disjoint sets of vertices such that $|X| = n - 4$.
- Let $\mathcal{L}$ be the family of graphs $G$ of order $n$ where $V(G) = \{x, y, u, v\} \cup X$ and with the following properties:
**Property 1**

$x$ and $y$ are non-adjacent, while $u$ and $v$ are adjacent.

**Figure:** A graph $L \in \mathcal{L}$
**Property 2**

Each vertex in \{x, y\} \cup X is adjacent to some vertex of \{u, v\}.

**Figure:** A graph \(L \in \mathcal{L}\)
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PROPERTY 3

Each vertex in \( \{u, v\} \cup X \) is non-adjacent to some vertex of \( \{x, y\} \).

\[ \]

**Figure:** A graph \( L \in \mathcal{L} \)
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**PROPERTY 4**
Each vertex in \( \{x, y\} \cup X \) is adjacent to some vertex of \( \{x, y\} \cup X \).

**Figure:** A graph \( L \in \mathcal{L} \)
**CHARACTERIZATION**

**PROPERTY 5**
Each vertex in \( \{u, v\} \cup X \) is non-adjacent to some vertex of \( \{u, v\} \cup X \).

**Figure**: A graph \( L \in \mathcal{L} \)
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**THEOREM**

(Hattingh, Jonck, Joubert and Plummer, 2008) Let $G$ be a graph of order $n \geq 2$ such that neither $G$ nor $\overline{G}$ contains isolated vertices or is isomorphic to $K$. Then

$$\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = 4 \text{ if and only if } G \in \mathcal{L}.$$
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**Figure:** L achieves the lower bound
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Figure: Shaded vertices are a TRDS
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\textbf{Figure: } \bar{L} achieves the lower bound
**LOWER BOUND EXAMPLE**

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**FIGURE**: Shaded vertices are a **TRDS**
Notice that $\gamma_{tr}(L) + \gamma_{tr}(\overline{L}) = 2 + 2 = 4$.

**Figure:** $L$ and $\overline{L}$
**Characterization Theorem**

**Description**

Let $\mathcal{U} = \{ G \mid G$ is a graph of order $n$ which can be obtained from a $P_4$ with consecutive vertices labeled $u, v_1, v_2, v$ by joining vertices $v_1$ and $v_2$ to each vertex of $K_{n-4}$ where $n \geq 6\}$. 

**Figure**: A graph $U \in \mathcal{U}$
THEOREM

(Hattingh, Jonck, Joubert and Plummer, 2008) Let $G$ be a graph of order $n$ such that neither $G$ nor $\overline{G}$ contains isolated vertices or is isomorphic to $K$. Then $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = n + 4$ if and only if $G \in \mathcal{U}$ or $\overline{G} \in \mathcal{U}$ or $G \cong P_4$. 
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Upper Bound Example: A graph achieving the upper bound
**Upper Bound Example**

**Figure**: Shaded vertices are a TRDS.
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**Figure:** A graph achieving the upper bound
**Upper Bound Example**

**Figure:** Shaded vertices are a **TRDS**
Notice that $\gamma_{tr}(U) + \gamma_{tr}(\bar{U}) = 4 + 6 = 4 + n(U)$.

**Figure:** $U$ and $\bar{U}$
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A Nordhaus-Gaddum Result for Total Restrained Domination

A Nordhaus-Gaddum Result for Restrained Domination

Partitions

Other recent papers
We obtain similar results for restrained domination.

Let $\mathcal{B} = \{P_3, \overline{P}_3\}$.

**Theorem**

(Domke, Hedetniemi, Laskar and Markus, 1999) Let $G = (V, E)$ be a graph of order $n \geq 2$ such that $G \notin \mathcal{B}$. Then $4 \leq \gamma_r(G) + \gamma_r(\overline{G}) \leq n + 2$.

Again, our proof is predicated on the level decomposition of a graph.
We obtain similar results for restrained domination.

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Theorem for Restrained Domination

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We again characterize the extremal graphs. However, there are now four types of graphs achieving the lower bound. We describe a graph of the first type.

DESCRIPTION OF TYPE 1 GRAPHS

- Let \( n \geq 5 \) be an integer and suppose \( \{x, y, z\} \) and \( X \) are disjoint sets of vertices such that \( |X| = n - 3 \).
- Let \( \mathcal{O} \) be the family of graphs \( G \) of order \( n \) where \( V(G) = \{x, y, z\} \cup X \) and with the following properties:
We again characterize the extremal graphs. However, there are now four types of graphs achieving the lower bound. We describe a graph of the first type.

**Description of Type 1 Graphs**

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**Description of Type 1 Graphs**

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- Let $\mathcal{O}$ be the family of graphs $G$ of order $n$ where $V(G) = \{x, y, z\} \cup X$ and with the following properties:
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Characterization

Property 1

x is adjacent to each vertex of \( \{y, z\} \cup X \).

\[ \text{Figure: A graph } J \in \mathcal{O} \]
**PROPERTY 2**

Each vertex of \(\{y, z\} \cup X\) is adjacent to some vertex of \(\{y, z\} \cup X\).

**FIGURE:** A graph \(J \in \mathcal{O}\)
**PROPERTY 3**

Each vertex of $X$ is non-adjacent to some vertex of $\{y, z\}$ and non-adjacent to some vertex in $X$.

**Figure**: A graph $J \in \mathcal{O}$
Let $\mathcal{H}$ be the union of the classes comprised of the four types of graphs.

**Theorem**

Let $G$ be a graph of order $n \geq 2$ such that $G \notin \mathcal{B}$. Then $\gamma_r(G) + \gamma_r(\overline{G}) = 4$ if and only if $G$ or $\overline{G} \in \mathcal{H}$. 

**Characterization Theorem**
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**Figure:** A Type 1 graph achieving the lower bound
**LOWER BOUND EXAMPLE**

**Figure**: Shaded vertices are a RDS
LOWER BOUND EXAMPLE

**Figure:** A Type 1 graph achieving the lower bound.
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Figure: Shaded vertices are a \textbf{RDS}
Notice that $\gamma_r(J) + \gamma_r(\overline{J}) = 1 + 3 = 4$.

**Figure:** $J$ and $\overline{J}$
CHARACTERIZATION

**Description**

- Let $G^* = \{ G \mid G \text{ or } \overline{G} \text{ is a galaxy of non-trivial stars} \}$.
- Let $S = \{ G \mid G \text{ or } \overline{G} \cong K_1 \cup S \text{ where } S \text{ is a star and } |S| \geq 3 \}$.
- Lastly, let $E = G^* \cup S$. 
CHARACTERIZATION

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- Let $G^* = \{ G \mid G$ or $\overline{G} \text{ is a galaxy of non-trivial stars} \}$.  
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- Lastly, let $\mathcal{E} = \mathcal{G}^* \cup \mathcal{S}$. 
THEOREM

(Hattingh, Jonck, Joubert and Plummer, 2008) Let $G = (V, E)$ be a graph of order $n \geq 2$ such that $G \notin B$. Then $\gamma_r(G) + \gamma_r(\overline{G}) = n + 2$ if and only if $G \in \mathcal{E}$. 

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**Upper Bound Example**

**Figure**: A graph achieving the upper bound
**Upper Bound Example**

**Figure:** Shaded vertices are a RDS.
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**Upper Bound Example**

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**Figure**: Shaded vertices are a RDS
**Upper Bound Example**

Notice that $\gamma_r(S) + \gamma_r(\bar{S}) = 7 + 2 = n(S) + 2$.

**Figure:** $S$ and $\bar{S}$
A classical result in domination theory is that if $S$ is a minimal dominating set of a graph $G$ without isolates, then $V - S$ is also a dominating set of $G$. Thus, the vertex set of every graph without any isolates can be partitioned into two dominating sets. However, it is not the case that the vertex set of every graph can be partitioned into two restrained dominating sets. For example, the vertex set of $C_5$ cannot be partitioned into two restrained dominating sets.
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**FIGURE: C_5**
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Figure: $C_5$
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Figure: $C_5$
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Figure: $C_5$
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\[ \gamma^{ex} \]

AND

Restrained Domination in Trees

Figure: \( C_5 \)
A partition of the vertex set can also be thought of as a coloring. In particular, a partition into two restrained dominating sets is a 2-coloring of the graph such that no vertex has a monochromatic neighborhood. As an example of such a 2-coloring in $K_n$ with $n \geq 4$, take any 2-coloring with at least two vertices of each color, while in $K_{m,n}$ with $m, n \geq 2$ take any 2-coloring where neither partite set is monochromatic.
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Figure: $C_5$ with an edge added
Zelinka showed that no minimum degree is sufficient to guarantee the existence of two restrained dominating sets.

In contrast, Calkin and Dankelmann and Feige et al. have shown that if the maximum degree is not too large relative to the minimum degree, then sufficiently large minimum degree guarantees arbitrarily many disjoint dominating sets, and hence taking union of pairs, arbitrarily many disjoint restrained dominating sets.
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We now consider the question of how many edges must be added to $G$ to ensure the partition of $V$ into two restrained dominating sets. We denote this minimum number by $rd(G)$. 
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![Diagram of a graph with an edge added]
We now consider the question of how many edges must be added to $G$ to ensure the partition of $V$ into two restrained dominating sets. We denote this minimum number by $rd(G)$. 

\begin{figure}
\centering
\includegraphics{figure.png}
\caption{C_5 with an edge added}
\end{figure}
PARTITIONS - RESULTS

THEOREM

(Broere, Dorfling, Goddard, Hattingh and Ungerer, 2004) \[ \text{If } T \text{ is a tree with } \ell \text{ leaves, then } \ell/2 \leq rd(T) \leq \ell/2 + 1. \]

THEOREM

(Goddard, Hattingh and Henning, 2005) \[ \text{If } G \text{ is a graph of order } n \geq 4 \text{ and minimum degree at least 2, then } \]
\[ rd(G) \leq (n - 2\sqrt{n})/4 + O(\log n), \text{ and that this bound is best possible.} \]
THEOREM

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Henning and Maritz proved the following results:

**Theorem**

If $G$ is a connected graph of order $n \geq 4$, maximum degree $\Delta$ where $\Delta \leq n - 2$, and minimum degree at least 2, then $\gamma_{tr}(G) \leq n - \Delta/2 - 1$; and this bound is sharp.

**Theorem**

If $G$ is a connected bipartite graph of order $n \geq 5$, maximum degree $\Delta$ where $3 \leq \Delta \leq n - 2$, and minimum degree at least 2, then $\gamma_{tr}(G) \leq n - \frac{2}{3} \Delta - \frac{2}{9} \sqrt{3\Delta} - 8 - \frac{7}{9}$; and this bound is sharp.
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*If G is a connected graph of order $n \geq 4$, maximum degree $\Delta$ where $\Delta \leq n - 2$, and minimum degree at least 2, then $\gamma_{tr}(G) \leq n - \Delta/2 - 1$; and this bound is sharp.*

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Hattingh and Plummer consider restrained bondage in graphs.

Raczek characterized the trees with equal restrained domination and total restrained domination numbers. Xue-gang Chen and Hong-yu Chen characterized the trees with equal total domination and total restrained domination numbers. Zelinka studied the restrained and total restrained domatic numbers of a graph.
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