Chaotic Behavior in a Deterministic Model of Manufacturing

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Imagine...

- A logistics system with tens of thousands of workers
- Operates at near optimality
- No management, no IT department, no industrial engineers, no consultants
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How should workers coordinate to share work?
“Bucket brigades”

Local operating rule

Work forward until someone takes your work; then go back and take work from a slower worker.
Bucket brigades
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Bucket brigades
Behavior of bucket brigades

**Theorem**

*Under bucket brigades, balance emerges spontaneously.*

In other words, the line *balances itself* — and better than any engineering department could do it!
Some users of bucket brigades

Anderson Merchandisers ( +25% )
Blockbuster Music ( +27% )
CVS Drugstores, Inc. ( +34% )
Dell Computer
Ford Parts Distribution Centers ( +50% )
The Gap ( +27% )

Harcourt-Brace
McGraw-Hill
Radio Shack
Readers Digest ( +8% )
Wawa ( +50% )
Walgreen’s
Order-picking in a warehouse
Assembling sandwiches
Assembling televisions

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2-worker bucket brigades

Time until next completion:

\[ t = \frac{1 - x}{v_2} \]

Distance travelled by worker 1 in that time:

\[ v_1 t \]

Location of next hand-off:

\[ f(x) = \frac{v_1}{v_2} (1 - x) \]
The dynamics function

\[ f(x) = \left( \frac{v_1}{v_2} \right) (1 - x). \]
Convergence to fixed point

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Undesirable self-organization
Convergence and chaos
Deterministic chaos

\[ x_{k+1} = f(x_k) \]

**Definition**

A map is **chaotic** iff there exists \( x_0 \) such that the orbit \( O(x_0) = \{x_0, x_1, \ldots\} \) is both dense and unstable in \([0, 1]\).
Chaotic dynamics

Worker 1 = \( (1, 1/3) \)

Worker 2 = \( (1, 1) \)
Chaotic dynamics
Chaotic dynamics
Chaotic dynamics
Chaotic dynamics
Sensitive dependence on initial conditions

\[ x^{(n+1)} = 1 - 2x^{(n)} \mod 1 \]

There is no least-significant bit!

0.110100
Sensitive dependence on initial conditions

\[ x^{(n+1)} = 1 - 2x^{(n)} \mod 1 \]

There is no least-significant bit!

0.110100
0.10100
0.10100?
Sensitive dependence on initial conditions

\[ x^{(n+1)} = 1 - 2x^{(n)} \mod 1 \]

There is no least-significant bit!

\[
\begin{align*}
0.11010000 \\
0.101000? \\
0.0100? \\
0.0100? \\
\end{align*}
\]
Sensitive dependence on initial conditions

\[ x^{(n+1)} = 1 - 2x^{(n)} \mod 1 \]

There is no least-significant bit!

0.110100
0.10100?
0.0100??
0.100???
Sensitive dependence on initial conditions

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\[
\begin{align*}
0.110100 & \quad ? \\
0.101000 & \quad ? \\
0.0100 & \quad ? \\
0.100 & \quad ? \\
0.0 & \quad ?
\end{align*}
\]
Sensitive dependence on initial conditions

\[ x^{(n+1)} = 1 - 2x^{(n)} \mod 1 \]

There is no least-significant bit!

\[
\begin{array}{ccccccc}
0.1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0.1 & 0 & 1 & 0 & 0 & ? & ? \\
0.0 & 1 & 0 & 0 & ? & ? & ? \\
0.1 & 0 & 0 & ? & ? & ? & ? \\
0.0 & 0 & ? & ? & ? & ? & ? \\
\end{array}
\]
Sensitive dependence on initial conditions

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\[
\begin{align*}
0.1 & 1 0 1 0 0 0 \\
0.1 & 0 1 0 0 0 ? \\
0.0 & 1 0 0 ? ? ? \\
0.1 & 0 0 ? ? ? ? \\
0.0 & 0 ? ? ? ? ? \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{align*}
\]
Symptoms of chaos

- Sensitive dependence on initial conditions
- Periodic orbits are unstable
- Dense orbits
- Cannot be reliably simulated
- Seemingly random behavior
Seemingly random start/intercompletion times

Cumulative Percentage vs. Intercompletion Time

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Strange attractors

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Variability

- Process variability
  - Fluctuations in processing time
  - Unforeseen outages
  - Setups
  - Worker availability
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- Flow variability
  - Interarrival time of work
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- Flow variability
  - Interarrival time of work

- Deterministic chaos
For more information

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