Overlap Number of Graphs

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Joint work with
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Representation of Graphs

**Idea** Assign each vertex $\nu$ a set $f(\nu)$ so that $uv \in E(G)$ iff $f(u)$ and $f(\nu)$ satisfy a natural relation. Use sets $f(\nu)$ in a natural class.
Representation of Graphs

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interval graph: each $f(v)$ is an interval (> 600 items)
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**Def.** finite representation: each $f(v)$ is finite
size of repn: $|f| = \left| \bigcup_{v \in V(G)} f(v) \right|$ 
inters. number $\theta_1(G) = \min$ size of finite inters. repn
overlap number $\varphi(G) = \min$ size of finite overlap repn
Overlap vs. Intersection

**Thm.** (Erdős–Goodman–Pósa [1966]) $\theta_1(G)$ is the min size of a decomposition of $G$ into complete subgraphs.

**Cor.** $\theta_1(G) = |E(G)|$ when $G$ has no triangles.
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1 12 23 34 4
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**Ex.** $\theta_1(K_{r,r}) = r^2$, but $\phi(K_{r,r}) = 3$. 
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\[
\begin{align*}
1 & \quad 12 & \quad 23 & \quad 34 & \quad 4 \\
01 & \quad 12 & \quad 23 & \quad 34 & \quad 0123
\end{align*}
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Results \((n\text{-vertex graphs})\)

**Rosgen** (masters’ thesis)

Caterpillars: \(\varphi(G) = \) order of the longest path.
Trees: \(\varphi(G) \leq n + 1.\)
Chordal Graphs: \(\varphi(G) \leq 2n.\)
Planar Graphs: \(\varphi(G) \leq (10/3)n - 6.\)
All Graphs: \(\varphi(G) \leq \lfloor n^2/4 \rfloor + n.\)
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**New Results**

Trees: linear algorithm, $\phi(G) =$ order of the skeleton.
Planar Graphs: $\phi(G) \leq 2n - 5$ for $n \geq 5$, sharp.
All Graphs: $\phi(G) \leq \frac{n^2}{4} - \frac{n}{2} - 1$ (large even $n$), sharp.
Edge Bounds: $\phi(G) \leq |E(G)| - 1$ (sharp for triangle-free connected graphs without star-cutsets).
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- $f(v_i)$ is minimal among all sets containing $i$. 
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**Skeleton** of $T$: keep the derived tree $T'$ plus one leaf neighbor of each leaf of $T'$. 
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Update: Add the new labels to all old sets containing $i$. 
Sketch of Lower Bound

Suffices to prove that if $T$ is a skeleton, then $\varphi(T) = n$.

**Idea** Induct on $n$. Given overlap repn $f$ for a skeleton $T$, find leaf $x$ [or leaf $x$ and nbr $v$] whose deletion yields a skeleton $T'$ for which $f - \{a\}$ [or $f - \{a, b\}$] is an overlap repn. This needs $|V(T')|$ elements, so $|f| \geq n$. 
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**Lem.** If $f$ is overlap repn of $G$, then $f - S$ is an overlap repn of $G$ iff $S$ does not contain $f(u) \cap f(v)$ or $f(u) - f(v)$ or $f(v) - f(u)$ for any edge $uv$. 
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**Def.** A set $S$ is $f$-uniform if every $f(v) \cap S$ is $S$ or $\varnothing$. [Proper subsets of an $f$-uniform set can be deleted!]
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**Lem.** If $f$ is overlap repn of $G$, and $N(v)$ is independent and contains no leaves, and $\{a, b\}$ is $f$-uniform except at $v$, then $f - \{a\}$ or $f - \{b\}$ is an overlap repn of $G$. 
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**Def.** \( x \) is minimal if \( f(x) \) contains no other \( f(\nu) \). Leaf \( x \) is doubly minimal if \( x \) and nbr \( \nu \) both minimal.

Find a doubly-minimal leaf \( x \).
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Case 1: \( d(u) = 2 \). \( G \) is a skeleton. Choose \( a, b \in f(x) \).
\( x \) minimal \( \Rightarrow \) \( f(x) \) is uniform except at \( \nu \).
Lemma makes \( f - \{a\} \) or \( f - \{b\} \) an overlap repn of \( G \).
Sketch of Lower Bound, continued

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Find a doubly-minimal leaf $x$.

Case 2: $d(u) > 2$. $G - v$ is a skeleton. Show that $f - \{a, b\}$ restricts to an overlap repn of $G - v$, where $a \in f(x) - f(v)$ and $b \in f(x) \cap f(v)$.
A Tool for Upper Bounds

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Decomposition $\Rightarrow |f(u) \cap f(\nu)| \leq 1$.

$\delta(G) \geq k \Rightarrow |f(\nu)| \geq 2 \Rightarrow$ no containments. □
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**Cor.** $\delta(G) \geq 2 \Rightarrow \varphi(G) \leq \Phi(G) \leq |E(G)|$. 


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**Cor.** $\delta(G) \geq 2 \Rightarrow \varphi(G) \leq \Phi(G) \leq |E(G)|$.

**Lem.** If $d(\nu) \leq 2$, then $\Phi(G) \leq \Phi(G - \nu) + 2$.

**Pf.** For $d(\nu) = 2$, let $f(\nu) = \{a, b\}$, add each to one nbr.
Upper Bounds for Planar Graphs

**Lem.** Every $n$-vertex planar graph, with $n \geq 3$, decomposes into at most $2n - 4$ edges and triangles.
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Pf. Induct on \# facial triangles. If none, $|E(G)| \leq 2n - 4$. Otherwise, let $[x, y, z]$ be a facial triangle. Form $G'$.
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$G'$ decomposes into $2n - 2$ pieces, using $vx, vy, vz$. Replace with $[x, y, z]$ to decompose $G$ into $2n - 4$. 

\[ \begin{array}{c}
\text{G} \\
\begin{array}{c}
x \\
y \\
z
\end{array}
\end{array} \quad \rightarrow \quad \begin{array}{c}
\text{G'} \\
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x \\
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$G'$ decomposes into $2n - 2$ pieces, using $vx, vy, vz$. Replace with $[x, y, z]$ to decompose $G$ into $2n - 4$.

**Cor.** If $G$ is planar and $\delta(G) \geq 3$, then $\Phi(G) \leq 2n - 4$. 

\[ 
\begin{align*} 
\text{G} & \quad \rightarrow \quad \text{G}' \\
\text{z} & \quad \text{x} \quad \text{y} \\
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\[ G \rightarrow G' \]

\( G' \) decomposes into \( 2n - 2 \) pieces, using \( vx, vy, vz \). Replace with \([x, y, z]\) to decompose \( G \) into \( 2n - 4 \).

**Cor.** If \( G \) is planar and \( \delta(G) \geq 3 \), then \( \Phi(G) \leq 2n - 4 \).

**Cor.** If \( G \) is planar and not \( K_{1,n-1} \), then \( \Phi(G) \leq 2n - 4 \).
The Extremal Result for Planar Graphs

**Thm.** (much more effort, not yet fully written)
For planar $G$ with $n$ vertices, $\varphi(G) \leq 2n - 5$ (if $n \geq 5$).
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The theorem is sharp when $4 \mid n$, as shown by $P_k \Box C_4$. 
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The theorem is sharp when $4 \mid n$, as shown by $P_k \square C_4$.

**Thm.** If $G$ is triangle-free and connected and has no star-cutset, then $\varphi(G) = |E(G)| - 1$.

**Def.** A **star-cutset** of a graph $G$ is a separating set $S$ such that $G[S]$ has a spanning star (some vertex of $S$ is adjacent to the rest of $S$).
Upper Bound Using $|E(G)|$

**Thm.** If $\delta(G) \geq 2$ and $n \geq 4$, then $\varphi(G) \leq |E(G)| - 1$. 
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**Pf.** Fix an edge $uv$. Define $f$ using an element for each edge other than $uv$.
For $w \notin \{u, v\}$, let $f(w) = \{ e \in E(G) : w \in e \}$.
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For $w \in \{u, v\}$, let $f(w) = \{e \in E(G) : w \notin e\}$.

For $w \notin \{u, v\}$, $f$ restricts to a pure overlap repn, since $\delta(G) \geq 2$ prohibits containments. Also, $f(u)$ and $f(v)$ overlap the sets for their nbrs (except in $K_2 \vee \overline{K}_{n-2}$).
Finally, $f(u)$ contains the labels for all edges incident to its nonneighbors.
Lower Bound Lemmas

\textbf{Lem.} If \(x\) and \(y\) are adjacent to each other but not to \(v\), then \(f(v)\) contains both or neither of \(\{f(x), f(y)\}\).
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$xy \in E(G) \Rightarrow f(y)$ intersects $f(x)$ and hence also $f(v)$. 

$\therefore f(y) \subseteq f(v)$ to prevent $yv \in E(G)$. 

\[ v \bullet \quad x \bullet \longrightarrow \bullet \ y \]
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\( xy \in E(G) \Rightarrow f(y) \) intersects \( f(x) \) and hence also \( f(v) \).
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\begin{array}{c}
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\end{array} \quad \begin{array}{c}
x \bullet \quad y \\
\end{array}
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**Lem.** If \( u \) and \( v \) are non-minimal for an overlap repn \( f \) of a connected \( G \) without star-cutsets, then \( uv \in E(G) \).
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\( \therefore f(y) \subseteq f(v) \) to prevent \( yv \in E(G) \). 

\[
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
\]

**Lem.** If \( u \) and \( v \) are non-minimal for an overlap repn \( f \) of a connected \( G \) without star-cutsets, then \( uv \in E(G) \).

**Pf.** \( uv \notin E(G) \Rightarrow v \in V(G) - N[u] \).
No star-cutset \( \Rightarrow G - N[u] \) connected.
\( u \) non-minimal \( \Rightarrow f(u) \supset f(v) \). Similarly, \( f(v) \supset f(u) \).

\[
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
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The Lower Bound

**Thm.** If $G$ is triangle-free, connected, and without star-cutsets, then $\varphi(G) = |E(G)| - 1$. 
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These labels are used for vertices outside $N(u)$ (tri-free) and hence lie in $f(u)$, since $f(u)$ is nonminimal.

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Case 2: two. This case is similar; one element can be saved. The configuration of the upper bound is forced.
Cor. If $G$ is the graph obtained by deleting a perfect matching from $K_{n/2,n/2}$, then $\varphi(G) = \frac{n^2}{4} - \frac{n}{2} - 1$. 
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Upper bound: Rosgen observed that \( \varphi(G) \leq \frac{n^2}{4} + n \).
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Improvement: $\varphi(G) \leq \frac{n^2}{4}$ using Erdős–Goodman–Pósa decomposition of $G$ into $\leq \frac{n^2}{4}$ edges and triangles.
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Better still: $\varphi(G) \leq \frac{n^2}{4} - \frac{n}{2} - 1$ for large enough even $n$. That is, the construction above is extremal (much more work, not yet all written down).
Sketch of the Better bound

**Thm.** \( \varphi(G) \leq \frac{n^2}{4} - \frac{n}{2} + 4 \) except for small \( n \).
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**Thm.** \( \phi(G) \leq \frac{n^2}{4} - \frac{n}{2} + 4 \) except for small \( n \).

**Pf.** If \( d(v) \leq 1 \), then \( \phi(G) \leq \phi(G - v) + 2 \).
Sketch of the Better bound

**Thm.** $\varphi(G) \leq \frac{n^2}{4} - \frac{n}{2} + 4$ except for small $n$.

**Pf.** If $d(\nu) \leq 1$, then $\varphi(G) \leq \varphi(G - \nu) + 2$.

If $G$ has a triangle $uvw$, use $\Phi(G - \{u, \nu, w\}) \leq \frac{(n-3)^2}{4}$. Add labels $f(u) = 12$, $f(\nu) = 23$, $f(w) = 31$. For other $x$, give a label to $x$ and to $N(x) \cap \{u, \nu, w\}$. 

$\frac{(n-3)^2}{4} + 3 + (n - 3) \leq \frac{n^2}{4} - \frac{n}{2} + 3$. 
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If \( G \) is triangle-free nonbipartite, \( |E(G)| \leq \frac{n^2}{4} - \frac{n}{2} + 4 \).

Also \( \delta(G) \geq 2 \); use \( \Phi(G) \leq |E(G)| \).
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If \( G \) is bipartite with two vertices having the same neighborhood, then \( \varphi(G) \leq \frac{(n-1)^2}{4} - \frac{n-1}{2} + 4 \).
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If \( G \) is bipartite with no repeated nbhd, an edge must be deleted from all but one vertex of the larger part.

\( \delta(G) \geq 2 \), so \( \Phi(G) \leq |E(G)| \leq (k - 1)(n - k) \leq \frac{(n-1)^2}{4} \).
Open Problems

1) What is the complexity of computing $\varphi(G)$? Presumably $\varphi(G) \leq k$ is NP-hard on general graphs, but it is not yet proved. What about planar graphs?
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4) How can one get the paper finished? (7 authors.)