The global attractivity and periodic character of some rational difference equations of arbitrary order

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• This is joint work with John D. Foley (UCSD) and Stevo Stević (Mathematical Institute of the Serbian Academy of Science)
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The Global Attractivity of the Rational Difference Equation $y_n = 1 + \frac{y_{n-k}}{y_{n-m}}$

by Kenneth S. Berenhaut, John D. Foley, and Stevo Stević

To appear in *Proceedings of the American Mathematical Society*
This paper studies the behavior of positive solutions of the recursive equation

\[ y_n = 1 + \frac{y_{n-k}}{y_{n-m}}, \quad n = 0, 1, \ldots, \quad (1) \]

with \( y_{-s}, y_{-s+1}, \ldots, y_{-1} \in (0, \infty) \) and \( k, m \in \{1, 2, 3, 4, \ldots\} \), where \( s = \max\{k, m\} \).

“In this paper the authors study the global stability, the boundedness, and the periodic nature of the positive solutions of the difference equation (1) $x_{n+1} = \alpha + x_{n-1}/x_n$, where $\alpha \in (0, \infty)$, and where the initial conditions $x_{-1}$ and $x_0$ are arbitrary positive real numbers. The authors show that a necessary and sufficient condition that every positive solution of (1) be bounded is $\alpha \geq 1$. Moreover, they show that if $\alpha = 1$ then every positive solution of (1) converges to a two-cycle, while if $\alpha > 1$, then $\overline{x} = \alpha + 1$ is a globally asymptotically stable equilibrium of equation (1).”

“We prove that every positive solution of the difference equation

\[ x_n = 1 + \frac{x_{n-2}}{x_{n-3}}, \quad n = 0, 1, \ldots, \]

converges to a period two solution.”
$k = 2, \ m = 3$

\begin{equation}
y_n = 1 + \frac{y_{n-2}}{y_{n-3}}.
\end{equation}

Figure 1: $y_n = 1 + \frac{y_{n-2}}{y_{n-3}}$. 
Figure 2: $y_n = 1 + y_{n-8}/y_{n-5}$. 
\[ k = 8, \quad m = 5 \]

Figure 3: \( y_n = 1 + y_{n-8}/y_{n-5}, \quad 1 \leq n \leq 100. \)

Figure 4: \( y_n = 1 + y_{n-8}/y_{n-5}, \quad 100 \leq n \leq 200. \)

Figure 5: \( y_n = 1 + y_{n-8}/y_{n-5}, \quad 200 \leq n \leq 300. \)
$k$ even theorem


**Theorem 1** *(Grove and Ladas)* Suppose that \( \gcd(m, k) = 1 \) with \( k \geq 2 \) even and \( m \geq 1 \) odd. Then every positive solution of (1) converges to a non-negative solution of (1) with period 2.
Figure 6: \( y_n = 1 + y_{n-7}/y_{n-3} \).

Figure 7: \( y_n = 1 + y_{n-9}/y_{n-5} \).

Figure 8: \( y_n = 1 + y_{n-9}/y_{n-5} \).
$k = 3, \ m = 4$

Figure 9: $y_n = 1 + y_{n-3}/y_{n-4},\ 4 \leq n \leq 30.$
\[ k = 4, \ m = 3 \]

Figure 10: \( y_n = 1 + \frac{y_{n-4}}{y_{n-3}}, \ 1 \leq n \leq 50. \)
$k$ odd theorem

Theorem 2 Suppose that $\gcd(m, k) = 1$ and that $\{y_i\}$ satisfies (1) with $y_{-s}, y_{-s+1}, \ldots, y_{-1} \in (0, \infty)$ where $s = \max\{m, k\}$. Then, if $k$ is odd, the sequence $\{y_i\}$ converges to the unique equilibrium 2.
The Proof for $k$ Odd

1 Preliminaries and Notation

Since the case $k = m$ is trivial, we will assume, throughout, that $k \neq m$.

- Consider the transformed sequence $\{y^*_i\}$ defined by

$$y^*_i = \begin{cases} 
  y_i, & \text{if } y_i \geq 2 \\
  3 - \frac{1}{y_i - 1}, & \text{otherwise}
\end{cases},$$

for $i \geq 0$. (2)
Distance in the transformed space

• Define the sequence $\{\delta_i\}$ via $\delta_i = |2 - y_i^*|$ for $i \geq 0$, so that

$$\delta_i = \begin{cases} |2 - y_i|, & \text{if } y_i \geq 2 \\ \frac{|2 - y_i|}{y_i - 1}, & \text{otherwise} \end{cases}.$$  

(3)
Contraction Lemma

Lemma 1  We have

\[ \delta_n \leq \max\{\delta_{n-k}, \delta_{n-m}\}, \]

for all \( n \geq s \).
Proof.

Suppose that \( \max\{\delta_{n-k}, \delta_{n-m}\} < \delta_n \)

If \( y_n > 2 \)

\( y_{n-k} < y_n \)

\( y_{n-m} \geq 2 \) gives \( y_{n-m} > \frac{1}{y_{n-1}} + 1 \)

\( y_{n-m} \leq 2 \) gives \( y_n - 2 > 2 - y_{n-m}^* = \frac{1}{y_{n-m-1}} - 1 \)

and hence \( y_{n-m} > \frac{1}{y_{n-1}} + 1 \).

Hence,

\[
y_n = 1 + \frac{y_{n-k}}{y_{n-m}} < 1 + \frac{y_n}{y_{n-1} + 1} = y_n. \tag{5}
\]
Similarly, if \( y_n < 2 \), we have

- \( y_{n-k} > y_n \)
- \( y_{n-m} < \frac{1}{y_{n-1}} + 1 \).

Hence,

\[
y_n = 1 + \frac{y_{n-k}}{y_{n-m}} > 1 + \frac{y_n}{\frac{1}{y_{n-1}} + 1} = y_n. \tag{6}
\]

In either case, we have a contradiction, and the lemma follows. \( \blacksquare \)
Now, set

\[ D_n = \max_{n-s \leq i \leq n-1} \{ \delta_i \}, \]  

(7)

for \( n \geq s \).

The following lemma is a simple consequence of Lemma 1.

\[ \star \]

**Lemma 2** The sequence \( \{D_i\} \) is monotonically non-increasing in \( i \), for \( i \geq s \).

Since \( D_i \geq 0 \) for \( i \geq s \), Lemma 2 implies that, as \( i \) tends to infinity, the sequence \( \{D_i\} \) converges to some limit, say \( D \), where \( D \geq 0 \).
2 Convergence of solutions

Proof of Theorem 2. Suffices to show that \( \{y_i^*\} \) converges to 2.

- By the definition, the values of \( D_i \) are taken on by entries in \( \{\delta_j\} \),
- By Lemma 1, \( y_i^* \in [2-D_i, 2+D_i] \) for \( i \geq s \).
  \[ \star \] Suppose \( D > 0 \).
- Then, for any \( \epsilon \in (0, D) \), we can find an \( N \) such that
  - \( y_N^* \in [2-D - \epsilon, 2-D + \epsilon] \) or \( y_N^* \in [2+D - \epsilon, 2+D + \epsilon] \) and
  - \( y_i^* \in [2-D - \epsilon, 2+D + \epsilon] \), for \( i \geq N - mk - s \).
If $y_N^* \in [2 + D - \epsilon, 2 + D + \epsilon]$

Suppose that $y_N^* \in [2 + D - \epsilon, 2 + D + \epsilon]$.

- Note that for $\epsilon$ sufficiently small, the hypotheses above guarantee that $y_{N-k} \geq 2$ and $y_{N-m} \leq 2$, where at least one of the inequalities is strict:
  - Suppose that $y_{N-k} \geq 2$ and $y_{N-m} \geq 2$. Then

$$y_N^* = 1 + \frac{y_{N-k}^*}{y_{N-m}^*} \leq 1 + \frac{2 + D + \epsilon}{2} = 2 + D - \epsilon - \left(\frac{D}{2} - \frac{3}{2} \epsilon\right)$$

$$< 2 + D - \epsilon$$

for $\epsilon$ sufficiently small, since $D > 0$. (Contradiction)
Suppose \( y_{N-k} \leq 2 \) and \( y_{N-m} \leq 2 \). Then

\[
y^*_N = 1 + \frac{1 + \frac{1}{3-y^*_{N-k}}}{1 + \frac{1}{3-y^*_{N-m}}} \leq 1 + \frac{2}{1 + \frac{1}{3-(2-D-\epsilon)}} = 1 + \frac{2}{1 + \frac{1}{1+D+\epsilon}}
\]

\[
= 2 + D - \epsilon - \frac{D^2 + D - (3\epsilon + \epsilon^2)}{2 + D + \epsilon} < 2 + D - \epsilon
\]

for \( \epsilon \) sufficiently small, since \( D > 0 \). (Contradiction)
\[ y_{N-k} \geq 2 \text{ and } y_{N-m} \leq 2 \]

\[
\therefore \text{ Thus, assume that } y_{N-k} \geq 2 \text{ and } y_{N-m} \leq 2.
\]

Solving for \( y^*_{N-k} \) and \( y^*_{N-m} \) in

\[
y^*_N = y_N = 1 + \frac{y_{N-k}}{y_{N-m}} = 1 + \frac{y^*_{N-k}}{1 + \frac{1}{3-y^*_{N-m}}},
\]

we have

\[
y^*_{N-k} = (y^*_N - 1) \left( 1 + \frac{1}{3-y^*_N} \right)
\]

and

\[
y^*_{N-m} = 3 - \frac{1}{\frac{y^*_{N-k}}{y^*_N - 1} - 1}
\]
Employing the inequalities $y_i^* \in [2 - D - \epsilon, 2 + D + \epsilon]$ and $y_N^* \in [2 + D - \epsilon, 2 + D + \epsilon]$, gives

$$2 + D + \epsilon \geq y_{N-k}^* \geq (1 + D - \epsilon) \left(1 + \frac{1}{3 - (2 - D - \epsilon)}\right)$$

$$= (1 + D - \epsilon) \left(\frac{2 + D + \epsilon}{1 + D + \epsilon}\right) = 2 + D - \epsilon \left(\frac{3 + D + \epsilon}{1 + D + \epsilon}\right)$$

$$\geq 2 + D - \epsilon \left(\frac{3 + D}{1 + D}\right)$$ \quad (13)$$

and

$$2 - D - \epsilon \leq y_{N-m}^* \leq 3 - \frac{1}{\frac{2 + D + \epsilon}{1 + D - \epsilon} - 1}$$

$$= 3 - \frac{1 + D - \epsilon}{1 + 2\epsilon} = 2 - D + \epsilon \left(\frac{3 + 2D}{1 + 2\epsilon}\right)$$

$$\leq 2 - D + \epsilon(3 + 2D).$$ \quad (14)$$
Thus

\[ 2 + D + \epsilon \left( \frac{3 + D}{1 + D} \right) \geq y^*_{N-k} \geq 2 + D - \epsilon \left( \frac{3 + D}{1 + D} \right) \]

(15)

and

\[ 2 - D - \epsilon(3 + 2D) \leq y^*_{N-m} \leq 2 - D + \epsilon(3 + 2D) \]
If \( y_N^* \in [2 - D - \epsilon, 2 - D + \epsilon] \)

Similarly when \( y_N^* \in [2 - D - \epsilon, 2 - D + \epsilon] \), \( y_{N-k} \leq 2 \) and \( y_{N-m} \geq 2 \), we have

\[
2 + D + \epsilon \left( \frac{3 + D}{1 + D} \right) \geq y_{N-m}^* \geq 2 + D - \epsilon \left( \frac{3 + D}{1 + D} \right) \tag{16}
\]

and

\[
2 - D - \epsilon(3 + 2D) \leq y_{N-k}^* \leq 2 - D + \epsilon(3 + 2D). \tag{17}
\]
Let $B = 3 + 2D > \frac{3 + D}{1 + D}$. Then, when $y_N^* \in [2 + D - \epsilon, 2 + D + \epsilon]$, iterating the above arguments gives

$$2 + D + \epsilon B \geq y_{N-k}^* \geq 2 + D - \epsilon B$$

$$2 + D + \epsilon B^2 \geq y_{N-2k}^* \geq 2 + D - \epsilon B^2$$

$$\vdots$$

$$\rightarrow 2 + D + \epsilon B^m \geq y_{N-mk}^* \geq 2 + D - \epsilon B^m$$ (18)

and,

$$2 - D - \epsilon B \leq y_{N-m}^* \leq 2 - D + \epsilon B$$

$$2 + D - \epsilon B^2 \leq y_{N-2m}^* \leq 2 + D + \epsilon B^2$$

$$\vdots$$

$$\rightarrow 2 + (-1)^k D - \epsilon B^k \leq y_{N-km}^* \leq 2 + (-1)^k D + \epsilon B^k.$$ (19)
Since $k$ is odd, (18) and (19) give that

$$y^*_{N-mk} \leq 2 - D + \epsilon B^k$$  \hfill (20)

and

$$y^*_{N-mk} \geq 2 + D - \epsilon B^m.$$  \hfill (21)

Thus, for sufficiently small $\epsilon$, we obtain a contradiction to the hypothesis that $D > 0$. A similar argument works when $y^*_N \in [2 - D - \epsilon, 2 - D + \epsilon]$, and the result is proven.
3  The periodic character of Equation (1)

Note that if $g = \gcd(m, k) > 1$ then $\{y_i\}$ can be separated into $g$ different equations of the form

$$y^{(j)}_n = 1 + \frac{y^{(j)}_{n-k}}{y^{(j)}_{n-m/g}},$$

(22)

where $j \in \{1, 2, \ldots, g\}$. Hence, we may assume that $\gcd(m, k) = 1$.

**Theorem 3**  Suppose that $\gcd(m, k) = 1$ with $k \geq 2$ even and $m \geq 1$ odd. Then every positive solution of (1) converges to a non-negative solution of (1) with period 2.

**Proof.** See Grove and Ladas (2005), Theorem 5.3.
The next theorem follows upon application of Theorems 2 and 3.

**Theorem 4** Suppose that $2^i \| k$ (i.e. $2^i$ is the largest power of 2 which divides $m$) and $2^j \| m$. Then, every solution of (1) converges to a period $t$ solution, where $t$ is given by

$$t = \begin{cases} 
1, & \text{if } j \geq i \\
2 \gcd(m, k), & \text{otherwise}
\end{cases} \quad (23)$$
Remark 2. Note that the argument used to prove Theorem 2 can be modified to show that in the case that \( \gcd(m, k) = 1 \) with \( k \) even, the period two solution for \( \{y^*_n\} \) is in fact of the form

\[
\ldots, 2 - D, 2 + D, 2 - D, 2 + D, \ldots, 
\]

where \( D \) is defined as in Section 2.
4 A variation on the theme: different constants and powers

The Global Attractivity of the Rational Difference Equation

\[ y_n = A + \left( \frac{y_{n-k}}{y_{n-m}} \right)^p \]

by Kenneth S. Berenhaut, John D. Foley, and Stevo Stević

in press for Proceedings of the American Mathematical Society
This paper studies the behavior of positive solutions of the recursive equation

\[ y_n = A + \left( \frac{y_{n-k}}{y_{n-m}} \right)^p, \quad n = 0, 1, \ldots, \quad (25) \]

with \( y_{-s}, y_{-s+1}, \ldots, y_{-1} \in (0, \infty) \) and \( k, m \in \{1, 2, 3, 4, \ldots\} \), where \( s = \max\{k, m\} \).

(The previous paper handled the case \( A = 1, \ p = 1 \).)
\(k = 3, \ m = 1, \ A < 1\)

Figure 11: \(y_n = A + y_{n-3}/y_{n-1}, \ A \in \{0.5, 0.414, 0.34, 0.1\}\).
\[ k = 3, \ m = 1, \ A < 1 \]

Figure 12: \[ y_n = A + y_{n-3}/y_{n-1}, \ A \in \{0.5, 0.414, 0.34, 0.1\}. \]
$k = 3, \ m = 1, \ A = .34$

Figure 13: \( y_n = .34 + \frac{y_{n-3}}{y_{n-1}}, \ 1 \leq n \leq 800 \).
Figure 14: $y_n = 0.34 + y_{n-3}/y_{n-1}$, $1 \leq n \leq 350$. 
A well known conjecture

Conjecture 1  If $A < \sqrt{2} - 1$, then all positive solutions to the equation

$$y_n = A + \frac{y_{n-3}}{y_{n-1}}$$

(26)

converge to the unique equilibrium $A + 1$. 
If $0 < A < 1$ and $0 < p \leq (A + 1)/2$

\*\*

**Theorem 5** Suppose that $m, k \geq 1$, and that $p, A$ are positive numbers satisfying $0 < A < 1$ and $0 < p \leq (A + 1)/2$. If the sequence $\{y_i\}$ satisfies (25) with $y_{-s}, y_{-s+1}, \ldots, y_{-1} \in (0, \infty)$ where $s = \max\{m, k\}$, then, $\{y_i\}$ converges to the unique equilibrium $A + 1$. 
Transformation

Set $z_n = y_n - A$, for $n \geq -s$. Then, equation (25) becomes

$z_n = \left( \frac{A + z_{n-k}}{A + z_{n-m}} \right)^p,$

(27)

for $n \geq 0$.

Now, define $\{z^*_i\}$ by

$z^*_i = \begin{cases} 
  z_i, & \text{if } z_i \geq 1 \\
  \frac{1}{z_i}, & \text{otherwise}
\end{cases}.$

(28)
Important Elementary Inequality

Lemma 3 \textit{If} $x > 1$ \textit{and} $0 < A < 1$, \textit{then}

\[
\left( \frac{A + x}{A + 1/x} \right)^{\frac{A+1}{2}} \leq x,
\]

\textit{with equality if and only if} $x = 1$. (29)
Proof.

- The inequality in (29) is equivalent to

\[
g_A(x) \overset{\text{def}}{=} (A + 1) \ln \left( \frac{A + x}{Ax + 1} \right) - (1 - A) \ln x \leq 0. \tag{30}
\]

- \( \lim_{x \to +\infty} g_A(x) = -\infty \) and \( g_A(1) = 0 \).

- \[
g'_A(x) = -\frac{A(x - 1)^2 (1 - A)}{(A + x)(Ax + 1)x} < 0, \tag{31}
\]

when \( x \neq 1 \), since \( A \in (0, 1) \).

Hence, \( g_A(x) \) is decreasing, and thus is negative on the interval \((1, \infty)\).  □

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Contraction

Lemma 4 Suppose \( \{ z_i \} \) satisfies (27) with \( p \leq (A + 1)/2 \) and \( A \in (0, 1] \). Then,

\[
1 \leq z^*_n \leq \max\{ z^*_{n-k}, z^*_{n-m}\},
\]

for all \( n \geq s \).
Proof.

If $z_{n-k} > z_{n-m}$

- Suppose that $z_{n-k} > z_{n-m}$
  - Set $x = \max\{z^*_{n-k}, z^*_{n-m}\}$.
  - If $z_{n-k} \geq 1$ then $1 \leq z_{n-k} \leq x$ and consequently
    \[ \frac{1}{x} \leq z_{n-k} \leq x, \] (33)
  - If $z_{n-k} < 1$, then $1/z_{n-k} = z^*_{n-k} \leq x$ from which (33) also holds.
- Similarly, we have that
  \[ \frac{1}{x} \leq z_{n-m} \leq x. \] (34)
Then, for $n \geq s$, we have that

$$z^*_n = z_n = \left( \frac{A + z_{n-k}}{A + z_{n-m}} \right)^p \leq \left( \frac{A + z_{n-k}}{A + z_{n-m}} \right)^{\frac{A+1}{2}} \leq \left( \frac{A + x}{A + \frac{1}{x}} \right)^{\frac{A+1}{2}} \leq x,$$

where the final inequality in (35) follows from the inequality lemma.

\textbf{If } z_{n-k} \leq z_{n-m} \textbf{ }

\textbf{★} Similarly, suppose $z_{n-k} \leq z_{n-m}$. Then

$$z^*_n = \frac{1}{z_n} = \left( \frac{A + z_{n-m}}{A + z_{n-k}} \right)^p \leq \left( \frac{A + z_{n-m}}{A + z_{n-k}} \right)^{\frac{A+1}{2}} \leq \left( \frac{A + x}{A + \frac{1}{x}} \right)^{\frac{A+1}{2}} \leq x.$$ 

\textbf{★}
Non-increasing bounds

Now, set

$$D_n = \max_{n-s \leq i \leq n-1} \{z_i^*\}, \quad (37)$$

for $n \geq s$.

The following result is a simple consequence of Lemma 4 and (37).

$\star$

**Lemma 5** The sequence $\{D_i\}$ is monotonically non-increasing in $i$, for $i \geq s$.

Since $D_i \geq 1$ for $i \geq s$, Lemma 5 implies that, as $i$ tends to infinity, the sequence $\{D_i\}$ converges to some limit, say $D$, where $D \geq 1$. 
5 Convergence of solutions

Showing $z^*_i \to 1 \ (D = 1)$

Proof of Theorem 5.

* Suffices to show that the transformed sequence $\{z^*_i\}$ converges to 1.

- For any $\epsilon > 0$, we can find an $N$ such that
  - $z^*_N \in [D, D+\epsilon]$,
  - $z^*_i \in [1, D+\epsilon]$, for $i \geq N - s$.

- Similar to before,
  \[
  \frac{1}{D + \epsilon} \leq z_{N-m}, z_{N-k} \leq D + \epsilon. \tag{38}
  \]
We will show that $D = 1$.

Suppose $D > 1$,

- Note that $z_N^* \in [D, D + \epsilon]$, implies that $z_N \neq 1$.

If $z_n > 1$

First, consider the case $z_N > 1$.

- Then, $z_N = z_N^* \in [D, D + \epsilon]$.

- Solving for $z_{n-k}$ in the definition, and employing the bounds, gives

$$D + \epsilon \geq z_{N-k} = z_N^{1/p} \left( A + z_{N-m} \right) - A$$

$$\geq D^{1/p} \left( A + \frac{1}{D + \epsilon} \right) - A$$

$$\geq D^{2/(k+1)} \left( A + \frac{1}{D + \epsilon} \right) - A. \quad (39)$$
This implies that

\[
\left( \frac{A + D + \epsilon}{A + \frac{1}{D+\epsilon}} \right) \geq D^{\frac{2}{A+1}}. 
\]

(40)
If $z_n < 1$

Assume now that $z_N < 1$.

- Then, $\frac{1}{z_N} = z^*_N \in [D, D + \epsilon]$
- From the equation for $\{z_i\}$ and the bounds on $z_{N-k}$ and $z_{N-m}$, it follows that

$$D + \epsilon \geq z_{N-m} = (z^*_N)^{1/p} (A + z_{N-k}) - A$$

$$\geq D^{1/p} \left( A + \frac{1}{D + \epsilon} \right) - A$$

$$\geq D^{\frac{2}{A+1}} \left( A + \frac{1}{D + \epsilon} \right) - A.$$ (41)

From (41) we have that (40) holds in this case, as well. Since $\epsilon > 0$ was arbitrary and $D > 1$, by Lemma 3 we arrive at a contradiction, which implies that $D = 1$, and the theorem follows.
Some difference equations with prime periodic solutions of high order

The Periodic Character of the Rational Difference Equation

\[ y_n = \frac{y_{n-m} + y_{n-m-k}}{y_{n-k}} \]

by John D. Foley and Kenneth S. Berenhaut

To appear in International Mathematical Forum
\[ y_n = \frac{(y_{n-7} + y_{n-9})}{y_{n-2}} \]
This paper studies the behavior of positive solutions of the recursive equation

\[ y_n = \frac{y_{n-m} + y_{n-m-k}}{y_{n-k}}, \quad n = 0, 1, \ldots, \]  \hspace{1cm} (42)

with \( y_{-s}, y_{-s+1}, \ldots, y_{-1} \in (0, \infty) \) and \( k, m \in \{1, 2, 3, 4, \ldots\} \), where \( s = k + m \).

● In Berenhaut, K. S., Dice, J. E., Foley, J. D., Iricanin, B. and Stevic, S., (2006) Periodic solutions of the rational difference equation \( y_n = \frac{y_{n-3} + y_{n-4}}{y_{n-1}} \), *J. Difference Equ. Appl.* 12, no. 2, 183–189, the authors proved that if \((k, m) = (1, 3)\), then every positive solution of (42) converges to a period two solution, answering Open Problem 11.4.8 (a) in Kulenović, M. R. S. and Ladas, G. *Dynamics of Second Order Rational Difference Equations. With open problems and conjectures.* Chapman and Hall/CRC, 2002.
Theorem for $y_n = \frac{y_{n-m} + y_{n-m-k}}{y_{n-k}}$

\[ \star \]

**Theorem 6**  If $\text{gcd}(m, 2k) = 1$, then every positive solution to the equation is asymptotically $2k$-periodic.
Boundedness character of positive solutions of a max difference equations

by Kenneth S. Berenhaut, John D. Foley, and Stevo Stević

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In this paper, we consider the case $c \in (0, 1)$ and $k$ odd for the equation

$$y_n = \max \left\{ c, \frac{y_{n-k}}{y_{n-m}} \right\}, \quad n \in \mathbb{N}_0,$$

(43)

where $k, m \in \mathbb{N}$.

- By using the change $y_n = 1/c^z_n$ the equation

becomes

$$z_n = \max\{-1, z_{n-k} - z_{n-m}\}, \quad n \in \mathbb{N}_0.$$

(44)
Periodicity for $z_i \in \mathbb{Z}$

\[\text{Theorem 7} \quad \text{Suppose that } \{z_n\} \text{ satisfies (44) with } z_{-s}, z_{-s+1}, \ldots, z_{-1} \in \mathbb{Z} \text{ where } s = \max\{m, k\}. \text{ Then, if } k \text{ is odd, the sequence } \{z_n\} \text{ converges to a periodic solution of the equation.}\]
We have the following conjecture regarding solutions to the equation.

\begin{equation}
\text{Conjecture 2 } \text{All positive solutions to the equation}
\end{equation}

\begin{equation}
z_n = \max\{-1, z_{n-k} - z_{n-m}\}
\end{equation}

with $k$ odd are eventually periodic.
Figure 15: $z_n = \max\{-1, z_{n-3} - z_{n-7}\} \ ( -1, 1, 10, -1, 2, 10, -1, 3, 9, -1, 4, 7, -1, 5, 4, -1, 6, 0, -1, 7, -1, -1, 8, -1, -1, 9, -1, 0, 10, \ldots )$. 
What are the periodic solutions to $z_n = \max\{-1, z_{n-k} - z_{n-m}\}$?

- Define the function $\sigma$ via

$$\sigma(n) \overset{\text{def}}{=} \max\{i \in \mathbb{N} : i(i - 1) \leq 2(n - 1)\} \quad (46)$$

for $n \in \mathbb{N}$.

**Lemma 6** We have

$$\sigma(n) = \left\lfloor \sqrt{2n} \right\rfloor \quad (47)$$

for $n \geq 2$, where $[x]$ indicates the nearest integer function.

We define the integer function $P$ via

$$P(m) \overset{\text{def}}{=} 2m + \sigma(m) = 2m + \left\lfloor \sqrt{2m} \right\rfloor. \quad (48)$$
High order prime periods for the equation
\[ z_n = \max\{-1, z_{n-k} - z_{n-m}\} \quad (k \text{ odd}) \]

\* \* 

**Theorem 8** For every \( k, m \geq 1 \), with \( \gcd(k, m) = 1 \), there exists a prime period \( P(m^*) = 2m^* + \lfloor \sqrt{2m^*} \rfloor \) solution for some \( m^* > (k - 1)^2 / 2 \).
Table 1: Some apparent prime periods for sufficiently small $A$, for the equation $y_n = A + y_{n-k}/y_{n-m}$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$m$</th>
<th>Apparent prime period</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>7</td>
<td>29</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
<td>40</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>22</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>24</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>31</td>
</tr>
<tr>
<td>7</td>
<td>4</td>
<td>57</td>
</tr>
<tr>
<td>11</td>
<td>17</td>
<td>166</td>
</tr>
</tbody>
</table>
Table 2: Prime periods of solutions to the equation $z_n = \max\{-1, z_{n-k} - z_{n-m}\}$ for various $k$ and $m$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$m$</th>
<th>$m^*$</th>
<th>prime periods ($P(m^*)$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>7</td>
<td>12</td>
<td>29</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
<td>17</td>
<td>40</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>9,11</td>
<td>22,27</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>10,12</td>
<td>24,29</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>13</td>
<td>31</td>
</tr>
<tr>
<td>7</td>
<td>4</td>
<td>25</td>
<td>57</td>
</tr>
<tr>
<td>11</td>
<td>17</td>
<td>77,82</td>
<td>166,177</td>
</tr>
</tbody>
</table>
\[ z_n = \frac{1}{1000} + \frac{z_{n-3}}{z_{n-7}} \]

Figure 16: \( z_n = \frac{1}{1000} + \frac{z_{n-3}}{z_{n-7}} \) (log base 1/A scale).
\[ \max\{-1, z_{n-3} - z_{n-7}\} \]

Figure 17: \( z_n = \max\{-1, z_{n-3} - z_{n-7}\} \)