Sampling of Continuous-Time Signals

Reference chapter 4 in Oppenheim and Schafer.
Periodic Sampling of Continuous Signals

\[ x[n] = x_c(nT), \quad -\infty < n < \infty. \]  \hspace{1cm} (4.1)

\[ f_s = 1/T \]

\[ \Omega_s = 2\pi/T \] when expressing frequencies in radians per second.

Figure 4.1  Block diagram representation of an ideal continuous-to-discrete-time (C/D) converter.
Mathematical Model for Periodic Sampling

\[ s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), \]  

(4.2)

Impulse train modulator followed by conversion of impulse train to sequence.

\[ x_s(t) = x_c(t)s(t) \]

\[ = x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x_c(t)\delta(t - nT). \]  

(4.3)

“sifting property”

\[ x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT), \]  

(4.4)
Mathematical Model for Periodic Sampling

C/D converter

Conversion from impulse train to discrete-time sequence

\[ x[n] = x_c(nT) \]
Mathematical Model for Periodic Sampling

Figure 4.2 Sampling with a periodic impulse train, followed by conversion to a discrete-time sequence. (a) Overall system. (b) $x_S(t)$ for two sampling rates. (c) The output sequence for the two different sampling rates.
Frequency-Domain Representation of Sampling

To derive the frequency-domain relation between the input and output of an ideal C/D converter, consider the Fourier transform of $x_s(t)$. Since, from Eq. (4.3), $x_s(t)$ is the product of $x_c(t)$ and $s(t)$, the Fourier transform of $x_s(t)$ is the convolution of the Fourier transforms $X_c(j\Omega)$ and $S(j\Omega)$ scaled by $\frac{1}{2\pi}$. The Fourier transform of the periodic impulse train $s(t)$ is the periodic impulse train

$$S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k \Omega_s),$$

where $\Omega_s = 2\pi / T$ is the sampling frequency in radians/s (see Oppenheim and Willsky, 1997 or McClellan, Schafer and Yoder, 2003). Since

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega),$$

where $*$ denotes the operation of continuous-variable convolution, it follows that

$$X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k \Omega_s)).$$
Figure 4.3  Frequency-domain representation of sampling in the time domain. (a) Spectrum of the original signal. (b) Fourier transform of the sampling function. (c) Fourier transform of the sampled signal with $\Omega_s > 2\Omega_N$. (d) Fourier transform of the sampled signal with $\Omega_s < 2\Omega_N$. 

aliasing (distortion)
Recovery of Signal

\[ X_r(j\Omega) = H_r(j\Omega)X_s(j\Omega), \]  

(4.8)

it follows that if \( H_r(j\Omega) \) is an ideal lowpass filter with gain \( T \) and cutoff frequency \( \Omega_c \) such that

\[ \Omega_N \leq \Omega_c \leq (\Omega_s - \Omega_N), \]

(4.9)

then

\[ X_r(j\Omega) = X_c(j\Omega), \]

(4.10)
Figure 4.4  Exact recovery of a continuous-time signal from its samples using an ideal lowpass filter.
Aliasing Example with Cosine

\[ x_c(t) = \cos \Omega_0 t, \]

\[ X_c(j\Omega) = \pi \delta(\Omega - \Omega_0) + \pi \delta(\Omega + \Omega_0) \]

no aliasing reconstruction:

\[ x_r(t) = \cos \Omega_0 t. \]

with aliasing reconstruction:

\[ x_r(t) = \cos(\Omega_s - \Omega_0)t; \]

(taken on identity, alias, of lower frequency signal)
Sampling Theorem

This discussion is the basis for the Nyquist sampling theorem (Nyquist 1928; Shannon, 1949), stated as follows.

**Nyquist-Shannon Sampling Theorem:** Let \( x_c(t) \) be a bandlimited signal with

\[
X_c(j\Omega) = 0 \quad \text{for } |\Omega| \geq \Omega_N. \tag{4.14a}
\]

Then \( x_c(t) \) is uniquely determined by its samples \( x[n] = x_c(nT) \), \( n = 0, \pm 1, \pm 2, \ldots \), if

\[
\Omega_s = \frac{2\pi}{T} \geq 2\Omega_N. \tag{4.14b}
\]

The frequency \( \Omega_N \) is commonly referred to as the *Nyquist frequency*, and the frequency \( 2\Omega_N \) as the *Nyquist rate*. 
General Case for Discrete-time Fourier Transform (DTFT)

\[ X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega Tn}. \]  

(4.15)

Since

\[ x[n] = x_c(nT) \]  

(4.16)

and

\[ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}, \]  

(4.17)

it follows that

\[ X_s(j\Omega) = X(e^{j\omega})|_{\omega=\Omega T} = X(e^{j\Omega T}). \]  

(4.18)

Consequently, from Eqs. (4.6) and (4.18),

\[ X(e^{j\Omega T}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)), \]  

(4.19)
or equivalently,

$$X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c \left[ j \left( \frac{\omega}{T} - \frac{2\pi k}{T} \right) \right].$$  \hspace{1cm} (4.20)

From Eqs. (4.18)–(4.20), we see that $X(e^{j\omega})$ is a frequency-scaled version of $X_s(j\Omega)$ with the frequency scaling specified by $\omega = \Omega T$. This scaling can alternatively be thought of as a normalization of the frequency axis so that the frequency $\Omega = \Omega_s$ in $X_s(j\Omega)$ is normalized to $\omega = 2\pi$ for $X(e^{j\omega})$. The frequency scaling or normalization in the
Examples of ambiguity due to sampling.

If we sample the continuous-time signal \( x_c(t) = \cos(4000\pi t) \) with sampling period \( T = 1/6000 \), we obtain \( x[n] = x_c(nT) = \cos(4000\pi T n) = \cos(\omega_0 n) \), where \( \omega_0 = 4000\pi T = 2\pi/3 \). In this case, \( \Omega_s = 2\pi/T = 12000\pi \), and the highest frequency of the signal is \( \Omega_0 = 4000\pi \), so the conditions of the Nyquist sampling theorem are satisfied and there is no aliasing. The Fourier transform of \( x_c(t) \) is

\[
X_c(j\Omega) = \pi \delta(\Omega - 4000\pi) + \pi \delta(\Omega + 4000\pi).
\]

Figure 4.6(a) shows

\[
X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c[j(\Omega - k \Omega_s)]
\]

(4.21)

\[
\omega_0 = 4000\pi T = 2\pi/3, \quad \omega_0 < \pi, \quad \Omega_0 = 4000\pi < \pi/T = 6000\pi
\]
Figure 4.6 (a) Continuous-time and (b) discrete-time Fourier transforms for sampled cosine signal with frequency $\Omega_0 = 4000\pi$ and sampling period $T = 1/6000$. 

(a) 

(b) 

$X(e^{j\omega}) = X_s(j\omega/T)$
Examples of ambiguity due to sampling.

First case with no aliasing.

\[ \omega_0 = 4000\pi \quad T = 2\pi / 3. \]
\[ \omega_0 < \pi \]
\[ \Omega_0 = 4000\pi < \pi / T = 6000\pi \]

Second case with aliasing.

Now suppose that the continuous-time signal is \( x_c(t) = \cos(16000\pi t) \), but the sampling period is \( T = 1/6000 \), as it was in Example 4.1. This sampling period fails to satisfy the Nyquist criterion, since \( \Omega_s = 2\pi / T = 12000\pi < 2\Omega_0 = 32000\pi \). Consequently, we expect to see aliasing. The Fourier transform \( X_s(j\Omega) \) for this case is identical to that of Figure 4.6(a). However, now the impulse located at \( \Omega = -4000\pi \) is from \( X_c[j(\Omega - \Omega_s)] \) in Eq. (4.21) rather than from \( X_c(j\Omega_s) \) and the impulse at \( \Omega = 4000\pi \) is from \( X_c[j(\Omega + \Omega_s)] \). That is, the frequencies \( \pm 4000\pi \) are alias frequencies. Plotting \( X(e^{j\omega}) = X_s(j\omega / T) \) as a function of \( \omega \) yields the same graph as shown in Figure 4.6(b), since we are normalizing by the same sampling period. The fundamental reason for this is that the sequence of samples is the same in both cases; i.e.,

\[ \cos(16000\pi n / 6000) = \cos(2\pi n + 4000\pi n / 6000) = \cos(2\pi n / 3). \]
Reconstruction of a band limited signal from its samples.

\[ x_s(t) = \sum_{n=-\infty}^{\infty} x[n] \delta(t - nT). \]  \hspace{1cm} (4.22)

\[ x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h_r(t - nT). \]  \hspace{1cm} (4.23)

\[ h_r(t) = \frac{\sin(\pi t/T)}{\pi t/T}. \]  \hspace{1cm} (4.24)

\[ x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}. \]  \hspace{1cm} (4.25)
Figure 4.7  (a) Block diagram of an ideal bandlimited signal reconstruction system. (b) Frequency response of an ideal reconstruction filter. (c) Impulse response of an ideal reconstruction filter.
Figure 4.8  Ideal bandlimited interpolation.
The properties of the ideal D/C converter are most easily seen in the frequency domain. To derive an input/output relation in this domain, consider the Fourier transform of Eq. (4.23) or Eq. (4.25), which is

\[ X_r(j\Omega) = \sum_{n=-\infty}^{\infty} x[n]H_r(j\Omega)e^{-j\Omega T n}. \]

Since \( H_r(j\Omega) \) is common to all the terms in the sum, we can write

\[ X_r(j\Omega) = H_r(j\Omega)X(e^{j\Omega T}). \]  (4.28)
Discrete-Time Processing of Continuous-Time Signals
Discrete-Time Processing of Continuous-Time Signals

The C/D converter produces a discrete-time signal

\[ x[n] = x_c(nT), \]  

(4.29)
i.e., a sequence of samples of the continuous-time input signal \( x_c(t) \). The DTFT of this sequence is related to the continuous-time Fourier transform of the continuous-time input signal by

\[ X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c \left[ j \left( \frac{\omega}{T} - \frac{2\pi k}{T} \right) \right]. \]  

(4.30)
The D/C converter creates a continuous-time output signal of the form

\[ y_r(t) = \sum_{n=-\infty}^{\infty} y[n] \frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}, \]  

(4.31)
where the sequence \( y[n] \) is the output of the discrete-time system when the input to the system is \( x[n] \). From Eq. (4.28), \( Y_r(j\Omega) \), the continuous-time Fourier transform of \( y_r(t) \), and \( Y(e^{j\omega}) \), the DTFT of \( y[n] \), are related by

\[ Y_r(j\Omega) = H_r(j\Omega)Y(e^{j\Omega T}) = \begin{cases} \frac{TY(e^{j\Omega T})}{\Omega}, & |\Omega| < \pi/T, \\ 0, & \text{otherwise}. \end{cases} \]  

(4.32)
Discrete-Time LTI Processing of Continuous-Time Signals

If the discrete-time system in Figure 4.10 is linear and time invariant, we have

\[ Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}), \quad (4.33) \]

where \( H(e^{j\omega}) \) is the frequency response of the system or, equivalently, the Fourier transform of the unit sample response, and \( X(e^{j\omega}) \) and \( Y(e^{j\omega}) \) are the Fourier transforms of the input and output, respectively. Combining Eqs. (4.32) and (4.33), we obtain

\[ Y_r(j\Omega) = H_r(j\Omega)H(e^{j\Omega T})X(e^{j\Omega T}). \quad (4.34) \]

Next, using Eq. (4.30) with \( \omega = \Omega T \), we have

\[ Y_r(j\Omega) = H_r(j\Omega)H(e^{j\Omega T})\frac{1}{T} \sum_{k=-\infty}^{\infty} X_c \left[ j \left( \Omega - \frac{2\pi k}{T} \right) \right]. \quad (4.35) \]
Discrete-Time LTI Processing of Continuous-Time Signals

If $X_c(j\Omega) = 0$ for $|\Omega| \geq \pi/T$, then the ideal lowpass reconstruction filter $H_r(j\Omega)$ cancels the factor $1/T$ and selects only the term in Eq. (4.35) for $k = 0$; i.e.,

$$Y_r(j\Omega) = \begin{cases} H(e^{j\Omega T})X_c(j\Omega), & |\Omega| < \pi/T, \\ 0, & |\Omega| \geq \pi/T. \end{cases}$$

(4.36)

Thus, if $X_c(j\Omega)$ is bandlimited and the sampling rate is at or above the Nyquist rate, the output is related to the input through an equation of the form

$$Y_r(j\Omega) = H_{\text{eff}}(j\Omega)X_c(j\Omega),$$

(4.37)

where

$$H_{\text{eff}}(j\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \pi/T, \\ 0, & |\Omega| \geq \pi/T. \end{cases}$$

(4.38)

That is, the overall continuous-time system is equivalent to an LTI system whose effective frequency response is given by Eq. (4.38).
Example 4.3  Ideal Continuous-Time Lowpass Filtering Using a Discrete-Time Lowpass Filter

Consider Figure 4.10, with the LTI discrete-time system having frequency response

\[ H(e^{j\omega}) = \begin{cases} 
1, & |\omega| < \omega_c, \\
0, & \omega_c < |\omega| \leq \pi.
\end{cases} \]  

(4.39)

This frequency response is periodic with period \(2\pi\), as shown in Figure 4.11(a). For bandlimited inputs sampled at or above the Nyquist rate, it follows from Eq. (4.38) that the overall system of Figure 4.10 will behave as an LTI continuous-time system with frequency response

\[ H_{\text{eff}}(j\Omega) = \begin{cases} 
1, & |\Omega T| < \omega_c \text{ or } |\Omega| < \omega_c / T, \\
0, & |\Omega T| \geq \omega_c \text{ or } |\Omega| \geq \omega_c / T.
\end{cases} \]  

(4.40)

As shown in Figure 4.11(b), this effective frequency response is that of an ideal lowpass filter with cutoff frequency \(\Omega_c = \omega_c / T\)
Figure 4.11  (a) Frequency response of discrete-time system in Figure 4.10. (b) Corresponding effective continuous-time frequency response for bandlimited inputs.
Figure 4.12  (a) Fourier transform of a bandlimited input signal. (b) Fourier transform of sampled input plotted as a function of continuous-time frequency $\Omega$. (c) Fourier transform $X(e^{j\omega})$ of sequence of samples and frequency response $H(e^{j\omega})$ of discrete-time system plotted versus $\omega$. (d) Fourier transform of output of discrete-time system. (e) Fourier transform of output of discrete-time system and frequency response of ideal reconstruction filter plotted versus $\Omega$. (f) Fourier transform of output.
Example 4.4  Discrete-Time Implementation of an Ideal Continuous-Time Bandlimited Differentiator

The ideal continuous-time differentiator system is defined by

$$y_c(t) = \frac{d}{dt} [x_c(t)],$$  \hspace{1cm} (4.43)

with corresponding frequency response

$$H_c(j\Omega) = j\Omega.$$  \hspace{1cm} (4.44)

Since we are considering a realization in the form of Figure 4.10, the inputs are restricted to be bandlimited. For processing bandlimited signals, it is sufficient that

$$H_{\text{eff}}(j\Omega) = \begin{cases} 
    j\Omega, & |\Omega| < \pi / T, \\
    0, & |\Omega| \geq \pi / T,
\end{cases}$$  \hspace{1cm} (4.45)

as depicted in Figure 4.13(a). The corresponding discrete-time system has frequency response

$$H(e^{j\omega}) = \frac{j\omega}{T}, \quad |\omega| < \pi,$$  \hspace{1cm} (4.46)
and is periodic with period $2\pi$. This frequency response is plotted in Figure 4.13(b). The corresponding impulse response can be shown to be

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \frac{j\omega}{T} \right) e^{j\omega n} d\omega = \frac{\pi n \cos \pi n - \sin \pi n}{\pi n^2 T}, \quad -\infty < n < \infty,$$

or equivalently,

$$h[n] = \begin{cases} 0, & n = 0, \\ \frac{\cos \pi n}{nT}, & n \neq 0. \end{cases}$$ (4.47)
Figure 4.13  (a) Frequency response of a continuous-time ideal bandlimited differentiator $H_c(j\Omega) = j\Omega$, $|\Omega| < \pi/T$. (b) Frequency response of a discrete-time filter to implement a continuous-time bandlimited differentiator.
4.4.2 Impulse Invariance

We have shown that the cascade system of Figure 4.10 can be equivalent to an LTI system for bandlimited input signals. Let us now assume that, as depicted in Figure 4.14, we are given a desired continuous-time system that we wish to implement in the form of Figure 4.10. With $H_c(j\Omega)$ bandlimited, Eq. (4.38) specifies how to choose $H(e^{j\omega})$ so that $H_{\text{eff}}(j\Omega) = H_c(j\Omega)$. Specifically,

$$H(e^{j\omega}) = H_c(j\omega/T), \quad |\omega| < \pi,$$

(4.48)

with the further requirement that $T$ be chosen such that

$$H_c(j\Omega) = 0, \quad |\Omega| \geq \pi/T.$$  

(4.49)

Under the constraints of Eqs. (4.48) and (4.49), there is also a straightforward and useful relationship between the continuous-time impulse response $h_c(t)$ and the discrete-time impulse response $h[n]$. In particular, as we shall verify shortly,

$$h[n] = Th_c(nT);$$

(4.50)

i.e., the impulse response of the discrete-time system is a scaled, sampled version of $h_c(t)$. When $h[n]$ and $h_c(t)$ are related through Eq. (4.50), the discrete-time system is said to be an impulse-invariant version of the continuous-time system.

Equation (4.50) is a direct consequence of the discussion in Section 4.2. Specifically, with $x[n]$ and $x_c(t)$ respectively replaced by $h[n]$ and $h_c(t)$ in Eq. (4.16), i.e.,

$$h[n] = h_c(nT),$$

(4.51)

Eq. (4.20) becomes

$$H(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_c \left( j \left( \frac{\omega}{T} - \frac{2\pi k}{T} \right) \right),$$

(4.52)
Figure 4.14 (a) Continuous-time LTI system. (b) Equivalent system for bandlimited inputs.
Example 4.5  A Discrete-Time Lowpass Filter Obtained by Impulse Invariance

Suppose that we wish to obtain an ideal lowpass discrete-time filter with cutoff frequency $\omega_c < \pi$. We can do this by sampling a continuous-time ideal lowpass filter with cutoff frequency $\Omega_c = \omega_c / T < \pi / T$ defined by

$$H_c(j\Omega) = \begin{cases} 1, & |\Omega| < \Omega_c, \\ 0, & |\Omega| \geq \Omega_c. \end{cases}$$

The impulse response of this continuous-time system is

$$h_c(t) = \frac{\sin(\Omega_c t)}{\pi t},$$

so we define the impulse response of the discrete-time system to be

$$h[n] = T h_c(nT) = T \frac{\sin(\Omega_c nT)}{\pi nT} = \frac{\sin(\omega_c n)}{\pi n},$$

where $\omega_c = \Omega_c T$. We have already shown that this sequence corresponds to the DTFT

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c, \\ 0, & \omega_c \leq |\omega| \leq \pi, \end{cases}$$

which is identical to $H_c(j\omega / T)$, as predicted by Eq. (4.55).
Continuous-Time Processing of Discrete-Time Signals

From the definition of the ideal D/C converter, \( X_c(j\Omega) \) and therefore also \( Y_c(j\Omega) \), will necessarily be zero for \(|\Omega| \geq \pi/T\). Thus, the C/D converter samples \( y_c(t) \) without aliasing, and we can express \( x_c(t) \) and \( y_c(t) \) respectively as

\[
x_c(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}
\]

and

\[
y_c(t) = \sum_{n=-\infty}^{\infty} y[n] \frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T},
\]

Figure 4.15  Continuous-time processing of discrete-time signals.
where \( x[n] = x_c(nT) \) and \( y[n] = y_c(nT) \). The frequency-domain relationships for Figure 4.15 are

\[
X_c(j\Omega) = TX(e^{j\Omega T}), \quad |\Omega| < \pi/T, \tag{4.58a}
\]

\[
Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega), \tag{4.58b}
\]

\[
Y(e^{j\omega}) = \frac{1}{T}Y_c\left(j\frac{\omega}{T}\right), \quad |\omega| < \pi. \tag{4.58c}
\]

Therefore, by substituting Eqs. (4.58a) and (4.58b) into Eq. (4.58c), it follows that the overall system behaves as a discrete-time system whose frequency response is

\[
H(e^{j\omega}) = H_c\left(j\frac{\omega}{T}\right), \quad |\omega| < \pi, \tag{4.59}
\]

or equivalently, the overall frequency response of the system in Figure 4.15 will be equal to a given \( H(e^{j\omega}) \) if the frequency response of the continuous-time system is

\[
H_c(j\Omega) = H(e^{j\Omega T}), \quad |\Omega| < \pi/T. \tag{4.60}
\]

Since \( X_c(j\Omega) = 0 \) for \( |\Omega| \geq \pi/T \), \( H_c(j\Omega) \) may be chosen arbitrarily above \( \pi/T \). A convenient—but arbitrary—choice is \( H_c(j\Omega) = 0 \) for \( |\Omega| \geq \pi/T \).
Example 4.7  Noninteger Delay

Let us consider a discrete-time system with frequency response

\[ H(e^{j\omega}) = e^{-j\omega\Delta}, \quad |\omega| < \pi. \]  \hfill (4.61)

When \( \Delta \) is an integer, this system has a straightforward interpretation as a delay of \( \Delta \), i.e.,

\[ y[n] = x[n - \Delta]. \]  \hfill (4.62)

When \( \Delta \) is not an integer, Eq. (4.62) has no formal meaning, because we cannot shift the sequence \( x[n] \) by a noninteger amount. However, with the use of the system of Figure 4.15, a useful time-domain interpretation can be applied to the system specified by Eq. (4.61). Let \( H_c(j\Omega) \) in Figure 4.15 be chosen to be

\[ H_c(j\Omega) = H(e^{j\Omega T}) = e^{-j\Omega T\Delta}. \]  \hfill (4.63)

Then, from Eq. (4.59), the overall discrete-time system in Figure 4.15 will have the frequency response given by Eq. (4.61), whether or not \( \Delta \) is an integer. To interpret the system of Eq. (4.61), we note that Eq. (4.63) represents a time delay of \( T\Delta \) seconds. Therefore,

\[ y_c(t) = x_c(t - T\Delta). \]  \hfill (4.64)

Furthermore, \( x_c(t) \) is the bandlimited interpolation of \( x[n] \), and \( y[n] \) is obtained by sampling \( y_c(t) \). For example, if \( \Delta = \frac{1}{2} \), \( y[n] \) would be the values of the bandlimited interpolation halfway between the input sequence values. This is illustrated in
Figure 4.16. We can also obtain a direct convolution representation for the system defined by Eq. (4.61). From Eqs. (4.64) and (4.56), we obtain

\[ y[n] = y_c(nT) = x_c(nT - T\Delta) \]

\[ = \sum_{k=-\infty}^{\infty} x[k]\frac{\sin[\pi(t - T\Delta - kT)/T]}{\pi(t - T\Delta - kT)/T}\bigg|_{t=nT} \]

\[ = \sum_{k=-\infty}^{\infty} x[k]\frac{\sin \pi(n - k - \Delta)}{\pi(n - k - \Delta)}, \tag{4.65} \]

which is, by definition, the convolution of \( x[n] \) with

\[ h[n] = \frac{\sin \pi(n - \Delta)}{\pi(n - \Delta)}, \quad -\infty < n < \infty. \]

When \( \Delta \) is not an integer, \( h[n] \) has infinite extent. However, when \( \Delta = n_0 \) is an integer, it is easily shown that \( h[n] = \delta[n-n_0] \), which is the impulse response of the ideal integer delay system.
Figure 4.16 (a) Continuous-time processing of the discrete-time sequence (b) can produce a new sequence with a “half-sample” delay.
Moving-Average Example

Figure 4.17 The moving-average system represented as a cascade of two systems.
Example 4.8  Moving-Average System with Noninteger Delay

In Example 2.16, we considered the general moving-average system and obtained its frequency response. For the case of the causal \((M + 1)\)-point moving-average system, \(M_1 = 0\) and \(M_2 = M\), and the frequency response is

\[
H(e^{j\omega}) = \frac{1}{(M + 1)} \frac{\sin[\omega(M + 1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}, \quad |\omega| < \pi.
\]  

(4.66)

This representation of the frequency response suggests the interpretation of the \((M + 1)\)-point moving-average system as the cascade of two systems, as indicated in Figure 4.17. The first system imposes a frequency-domain amplitude weighting. The second system represents the linear-phase term in Eq. (4.66). If \(M\) is an even integer (meaning the moving average of an odd number of samples), then the linear-phase term corresponds to an integer delay, i.e.,

\[
y[n] = w[n - M/2].
\]  

(4.67)
However, if $M$ is odd, the linear-phase term corresponds to a noninteger delay, specifically, an integer-plus-one-half sample interval. This noninteger delay can be interpreted in terms of the discussion in Example 4.7; i.e., $y[n]$ is equivalent to bandlimited interpolation of $w[n]$, followed by a continuous-time delay of $MT/2$ (where $T$ is the assumed, but arbitrary, sampling period associated with the D/C interpolation of $w[n]$), followed by C/D conversion again with sampling period $T$. This fractional delay is illustrated in Figure 4.18. Figure 4.18(a) shows a discrete-time sequence $x[n] = \cos(0.25\pi n)$. This sequence is the input to a six-point ($M = 5$) moving-average filter. In this example, the input is “turned on” far enough in the past so that the output consists only of the steady-state response for the time interval shown. Figure 4.18(b) shows the corresponding output sequence, which is given by

\[
y[n] = H(e^{j0.25\pi})\frac{1}{2}e^{j0.25\pi n} + H(e^{-j0.25\pi})\frac{1}{2}e^{-j0.25\pi n} \]

\[
= \frac{1}{2} \frac{\sin[3(0.25\pi)]}{6 \sin(0.125\pi)} e^{-j(0.25\pi)5/2} e^{j0.25\pi n} + \frac{1}{2} \frac{\sin[3(-0.25\pi)]}{6 \sin(-0.125\pi)} e^{j(0.25\pi)5/2} e^{-j0.25\pi n} \]

\[
= 0.308 \cos[0.25\pi(n - 2.5)].
\]

In this case works out to delay of 5/2 or 2.5 samples.
Figure 4.18 Illustration of moving-average filtering. (a) Input signal $x[n] = \cos(0.25\pi n)$. (b) Corresponding output of six-point moving-average filter.
Changing the Sampling Rate with Discrete-Time Processing

\[ x[n] = x_c(nT), \quad (4.68) \]

\[ x_1[n] = x_c(nT_1), \quad (4.69) \]

“resampling”
4.6.1 Sampling Rate Reduction by an Integer Factor

The sampling rate of a sequence can be reduced by “sampling” it, i.e., by defining a new sequence

\[ x_d[n] = x[nM] = x_c(nMT) . \]  

(4.70)

![Diagram showing sampling rate reduction](image)

**Sampling period** $T$

**Sampling period** $T_d = MT$

**Figure 4.19** Representation of a compressor or discrete-time sampler.

“downsampling”

“(sampling rate) compressor”
As in the case of sampling a continuous-time signal, it is useful to obtain a frequency-domain relation between the input and output of the compressor. This time, however, it will be a relationship between DTFTs. Although several methods can be used to derive the desired result, we will base our derivation on the results already obtained for sampling continuous-time signals. First, recall that the DTFT of \( x[n] = x_c(nT) \) is

\[
X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c \left[ j \left( \frac{\omega}{T} - \frac{2\pi k}{T} \right) \right].
\] (4.71)

Similarly, the DTFT of \( x_d[n] = x[nM] = x_c(nT_d) \) with \( T_d = MT \) is

\[
X_d(e^{j\omega}) = \frac{1}{T_d} \sum_{r=-\infty}^{\infty} X_c \left[ j \left( \frac{\omega}{T_d} - \frac{2\pi r}{T_d} \right) \right].
\] (4.72)

Now, since \( T_d = MT \), we can write Eq. (4.72) as

\[
X_d(e^{j\omega}) = \frac{1}{MT} \sum_{r=-\infty}^{\infty} X_c \left[ j \left( \frac{\omega}{MT} - \frac{2\pi r}{MT} \right) \right].
\] (4.73)

\[
r = i + kM,
\] (4.74)
where \( k \) and \( i \) are integers such that \(-\infty < k < \infty\) and \(0 \leq i \leq M - 1\). Clearly, \( r \) is still an integer ranging from \(-\infty\) to \(\infty\), but now Eq. (4.73) can be expressed as

\[
X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} \left\{ \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c \left[ j \left( \frac{\omega}{MT} - \frac{2\pi k}{T} - \frac{2\pi i}{MT} \right) \right] \right\}. \tag{4.75}
\]

The term inside the square brackets in Eq. (4.75) is recognized from Eq. (4.71) as

\[
X \left( e^{j(\omega-2\pi i)/M} \right) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c \left[ j \left( \frac{\omega - 2\pi i}{MT} - \frac{2\pi k}{T} \right) \right]. \tag{4.76}
\]

Thus, we can express Eq. (4.75) as

\[
X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} X \left( e^{j(\omega/M-2\pi i/M)} \right). \tag{4.77}
\]

\[
X \left( e^{j\omega} \right) = 0, \quad \omega_N \leq |\omega| \leq \pi, \tag{4.78}
\]

In this example, \(2\pi/T = 4\Omega_N\); i.e., the original sampling rate is exactly twice the minimum rate to avoid aliasing. Thus, when the original sampled sequence is downsampled by a factor of \(M = 2\), no aliasing results. If the downsampling factor is more than 2 in this case, aliasing will result, as illustrated in Figure 4.21.
Figure 4.20 Frequency-domain illustration of downsampling.

(a) Frequency-domain representation of downsampling.

(b) Frequency-domain representation with downsampling factor $M = 2$.

(c) Frequency-domain representation showing the relationship between the original and downsampled signals.

(d) Frequency-domain representation with the downsampled signal $X_d(e^{j\Omega T_d})$.

(e) Frequency-domain representation with the downsampled signal $X_d(e^{j\Omega T_d})$.
Figure 4.21  (a)–(c) Downsampling with aliasing. (d)–(f) Downsampling with prefiltering to avoid aliasing.
In general, to avoid aliasing in downsampling by a factor of $M$ requires that

$$\omega_N M \leq \pi \quad \text{or} \quad \omega_N \leq \frac{\pi}{M}. \quad (4.79)$$

*Figure 4.22* General system for sampling rate reduction by $M$. 

---

**Diagram Description:**
- **Input:** $x[n]$ with sampling period $T$.
- **First Block:** Lowpass filter with gain 1, cutoff $\pi/M$.
- **Second Block:** Downsampling by $M$, resulting in $\tilde{x}_d[n] = \tilde{x}[nM]$ with sampling period $T_d = MT$.
4.6.2 Increasing the Sampling Rate by an Integer Factor

We have seen that the reduction of the sampling rate of a discrete-time signal by an integer factor involves sampling the sequence in a manner analogous to sampling a continuous-time signal. Not surprisingly, increasing the sampling rate involves operations analogous to D/C conversion. To see this, consider a signal $x[n]$ whose sampling rate we wish to increase by a factor of $L$. If we consider the underlying continuous-time signal $x_c(t)$, the objective is to obtain samples

$$x_i[n] = x_c(nT_i),$$  \hspace{1cm} (4.80)

where $T_i = T/L$, from the sequence of samples

$$x[n] = x_c(nT).$$  \hspace{1cm} (4.81)

We will refer to the operation of increasing the sampling rate as *upsampling*.

From Eqs. (4.80) and (4.81), it follows that

$$x_i[n] = x[n/L] = x_c(nT/L), \hspace{1cm} n = 0, \pm L, \pm 2L, \ldots$$  \hspace{1cm} (4.82)

Figure 4.23 shows a system for obtaining $x_i[n]$ from $x[n]$ using only discrete-time processing. The system on the left is called a *sampling rate expander* (see Crochiere and Rabiner, 1983 and Vaidyanathan, 1993) or simply an *expander*. Its output is

$$x_e[n] = \begin{cases} 
  x[n/L], & n = 0, \pm L, \pm 2L, \ldots, \\
  0, & \text{otherwise},
\end{cases}$$  \hspace{1cm} (4.83)
or equivalently,

$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - kL].$$

(4.84)

The system on the right is a lowpass discrete-time filter with cutoff frequency $\pi/L$ and gain $L$. This system plays a role similar to the ideal D/C converter in Figure 4.9(b). First, we create a discrete-time impulse train $x_e[n]$, and we then use a lowpass filter to reconstruct the sequence.
The operation of the system in Figure 4.23 is most easily understood in the frequency domain. The Fourier transform of \( x_e[n] \) can be expressed as

\[
X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \left( \sum_{k=-\infty}^{\infty} x[k]\delta[n - kL] \right) e^{-j\omega n}
\]

\[
= \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega L k} = X(e^{j\omega L}).
\]  

(4.85)

Thus, the Fourier transform of the output of the expander is a frequency-scaled version of the Fourier transform of the input; i.e., \( \omega \) is replaced by \( \omega L \) so that \( \omega \) is now normalized by

\[
\omega = \Omega T_i.
\]  

(4.86)
As in the case of the D/C converter, it is possible to obtain an interpolation formula for \( x_i[n] \) in terms of \( x[n] \). First, note that the impulse response of the lowpass filter in Figure 4.23 is

\[
h_i[n] = \frac{\sin(\pi n/L)}{\pi n/L}. \tag{4.87}
\]

Using Eq. (4.84), we obtain

\[
x_i[n] = \sum_{k=-\infty}^{\infty} x[k] \frac{\sin[\pi(n-kL)/L]}{\pi(n-kL)/L}. \tag{4.88}
\]

The impulse response \( h_i[n] \) has the properties

\[
\begin{align*}
  h_i[0] &= 1, \\
  h_i[n] &= 0, \quad n = \pm L, \pm 2L, \ldots
\end{align*}\tag{4.89}
\]

Thus, for the ideal lowpass interpolation filter, we have

\[
x_i[n] = x[n/L] = x_c(nT/L) = x_c(nT_i), \quad n = 0, \pm L, \pm 2L, \ldots \tag{4.90}
\]

as desired. The fact that \( x_i[n] = x_c(nT_i) \) for all \( n \) follows from our frequency-domain argument.
Linear interpolation corresponds to interpolation so that the samples between two original samples lie on a straight line connecting the two original sample values. Linear interpolation can be accomplished with the system of Figure 4.23 with the filter having the triangularly shaped impulse response

\[ h_{\text{lin}}[n] = \begin{cases} 
1 - |n|/L, & |n| \leq L, \\
0, & \text{otherwise},
\end{cases} \]  
(4.91)

as shown in Figure 4.25 for \( L = 5 \). With this filter, the interpolated output will be

\[ x_{\text{lin}}[n] = \sum_{k=n-L+1}^{n+L-1} x_e[k] h_{\text{lin}}[n - k]. \]  
(4.92)

Figure 4.26(a) depicts \( x_e[k] \) (with the envelope of \( h_{\text{lin}}[n-k] \) shown dashed for a particular value \( n = 18 \)) and the corresponding output \( x_{\text{lin}}[n] \) for the case \( L = 5 \). In this case, \( x_{\text{lin}}[n] \) for \( n = 18 \) depends only on original samples \( x[3] \) and \( x[4] \). From this figure, we see that \( x_{\text{lin}}[n] \) is identical to the sequence obtained by connecting the two original samples on either side of \( n \) by a straight line and then resampling at the \( L - 1 \) desired points in
Figure 4.24  Frequency-domain illustration of interpolation.
Figure 4.25 Impulse response for linear interpolation.

\[ H_{\text{lin}}(e^{j\omega}) = \frac{1}{L} \left[ \frac{\sin(\omega L/2)}{\sin(\omega/2)} \right]^2. \] (4.93)
Figure 4.26  (a) Illustration of linear interpolation by filtering. (b) Frequency response of linear interpolator compared with ideal lowpass interpolation filter.
Figure 4.27  Illustration of interpolation involving $2K = 4$ samples when $L = 5$. 
Figure 4.28 Impulse responses and frequency responses for linear and cubic interpolation.
Figure 4.29  (a) System for changing the sampling rate by a noninteger factor. (b) Simplified system in which the decimation and interpolation filters are combined.
Figure 4.30 Illustration of changing the sampling rate by a noninteger factor.