The Albert Nonassociative Algebra System: A Progress Report

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Abstract

After four years of experience with the nonassociative algebra program **Albert**, we highlight its successes and drawbacks. Among its successes are the discovery of several new results in nonassociative algebra. Each of these results has been independently verified – either with a traditional mathematical proof or with an independent computation.

1 Introduction

By a *nonassociative algebra* we mean a vector space over a field, also having a multiplication operation that interacts with addition through the ordinary distributive laws. The multiplication operation, however, need not be commutative or associative. **Albert** is an interactive program to assist the specialist in the study of nonassociative algebras. Work on the system began in 1989. For more background, we refer the reader to the tutorial [13] and user guide [12].

Computers were used to study nonassociative algebras as early as the 1960's. The 1976 annual meeting of the American Mathematical Society in San Antonio held a special session on Computers in the Study of Nonassociative Rings and Algebras. The proceedings of this session [2] contain many computer applications by Kleinfeld, Hentzel, Luks, and others (e.g. [14], [16], [4]).

The main problem addressed by **Albert** is the recognition of *nonassociative polynomial identities*. A fundamental problem asks whether an algebra must satisfy a certain identity, assuming that it satisfies another given set of identities. For example, assume an algebra satisfies

$$\begin{aligned} (xy)z &= x(yz) \quad (1) \\ x^2 &= 0, \quad (2) \end{aligned}$$

for all x, y, z. Identity (1) is the associative law. From identity (2) one has $0 = (x+y)^2 = x^2 + y^2 + xy + yx$, getting $xy = -yx$. Together (1) and (2) imply

$$\begin{aligned} (xy)z &= -z(xy) \\ &= -(zx)y \\ &= +(xz)y \\ &= +x(zy) \\ &= -x(yz) \\ &= -(xy)z. \end{aligned}$$

Thus $2(xy)z = 0$, and, assuming a field of characteristic $\neq 2$, we have $(xy)z = 0$. The product of any three elements is zero.

This simple example serves only to illustrate the spirit of our problem domain. It is really a special case of a theorem by Nagata and Higman which states *if $\mathcal{A}$ is an associative algebra satisfying $x^n = 0$ for some n, and having no elements of additive order n, then the product of any $2^n - 1$ elements in $\mathcal{A}$ is zero.* ([23], p.126).

Experience seems to show that recognizing identities that are implied by a set of defining identities is inherently difficult. One successful approach by Hentzel uses group representation theory [4]. Another way to solve the problem, however, is to construct the *free* algebra determined by the given set of defining identities; if the polynomial in question is zero in the free algebra then it is an identity. This is the approach taken by **Albert**.

The reader should take note that we are dealing with identities (universally quantified equations) and not relations on generators. Thus, we are not asking if the target polynomial is a member of the *ideal* generated by the defining polynomials, but rather whether it is a member of the so-called *T-ideal* ([23], p.4).

Albert is a relatively small program with a simple user interface. It has no programming capability, and its problem domain is quite narrow compared with general purpose computer algebra systems like Maple, or automatic theorem provers like Isabelle [19]. **Albert** is interactive, and works in the following way. Suppose a person wishes to study *commutative algebras with nilindex 3.* That is, algebras satisfying

$$\begin{aligned} xy - yx &= 0 \quad (3) \\ x^3 &= 0. \quad (4) \end{aligned}$$

Suppose the researcher wishes to know if, in the presence of identities (3) and (4) the identity

$$(((a^2b^2)c^2)d^2)e = 0 \quad (5)$$

also holds. (Take note we are not assuming associativity. If we were, then the theorem of Nagata-Higman would imply that this expression is zero. Nevertheless, this is a reasonable conjecture, since we are assuming that multiplication is commutative.) To solve this using **Albert**, the user first enters identities (3) and (4):

```
identity xy - yx
identity (xx)x
```

Albert provides about a dozen operators that can be composed in arbitrary ways. One such operator is the *commutator* $[x,y]$, defined as $xy - yx$. **Albert** always interprets `x^n` to mean the particular product

$$(\ldots((x^2)x)x\ldots)x.$$

Other associations, like $x((xx)x)$, must be explicitly represented. Of course, in the presence of commutativity, x^3 could mean only one thing. Thus, the above two commands could also be expressed as:

```
identity [x,y]
identity x^3
```

Having supplied the defining identities, next the user supplies the degrees of the letters in the target polynomial (5) with:

```
generators 2a2b2c2d1e
```

To decide whether a given polynomial is an identity, **Albert** must first construct a sufficiently large homomorphic image of the free algebra. The above command means that the constructed algebra will include products in the generators whose degrees in a, b, c, d, e are at most 2,2,2,2,1 respectively. This is sufficient to decide the validity of (5). **Albert** does all work over a Galois field Z_p, where $2 \leq p < 256$. The user may specify the field by giving the prime p:

```
field 251
```

The user instructs **Albert** to begin the construction by typing:

```
build
```

This particular construction took about an hour of computer time, but other problems often require only seconds. Often, in practice, the construction fails due to the lack of sufficient memory or time. However, if the construction is successful, the user can query whether the polynomial is an identity by typing:

```
polynomial ( ( (a^2 b^2) c^2) d^2) e
```

In this case, **Albert** then responded with:

```
Polynomial is an identity.
```

This led to the following theorem:

Theorem 1 *Let $\mathcal{A}$ be a commutative algebra over a field having characteristic $\neq 2, 3$ and satisfying $x^3 = 0$. Then $\mathcal{A}$ satisfies*

$$(((a^2b^2)c^2)d^2)e = 0. \tag{6}$$

Theorem 1 has been independently confirmed by another computational method [8]. Although interesting in its own right, the result is critical to understanding the structure of Bernstein algebras [8]. In the remainder of this paper we shall identify some other successes of **Albert** in the last several years, as well as pinpoint some weaknesses.

The method employed for constructing the free algebra is a dynamic programming algorithm described in [9]. For a fixed set a defining identities, the algorithm runs in time that is polynomial in the total degree of the target polynomial *and* the dimension of the resulting algebra. This is somewhat misleading, since the dimension usually grows exponentially with the degree of the target polynomial. Hence the algorithm does not run in time polynomial in the total degree of the target polynomial.

2 A sufficient condition for commutativity

The identity

$$(xy)x - x(yx) = 0 \tag{7}$$

is known as the *flexible* law. It is customary in nonassociative algebra to use the ternary operator (x, y, z), called the *associator*, to denote the quantity $(xy)z - x(yz)$. The associator is available in **Albert**, and so flexibility may be expressed as:

```
identity (x,y,x)
```

Clearly flexibility is a generalization of associativity. Note that it is also implied by the commutative law as well, since commutativity would imply $(xy)x = x(xy) = x(yx)$. In general a flexible algebra need not be commutative. While experimenting with **Albert**, we observed that, for characteristic $\neq 2$, if the algebra was generated by a single element, then flexibility and commutativity seemed to be equivalent. This led to the following theorem, proven in [5].

Theorem 2 *Let $\mathcal{A}$ be an algebra over a field of characteristic $\neq 2$. Then $\mathcal{A}$ is commutative if and only if it is flexible and generated by a set S whose elements pairwise commute.*

Also given in [5] is a counterexample showing that "characteristic $\neq 2$" is necessary.

3 A strange identity of Chen

One of **Albert**'s most interesting discoveries concerns the identity

$$(xy)z = y(zx). \tag{8}$$

This identity seems to have been first studied by Chen in 1970 [3]. Obviously, any binary operation that is *both* commutative and associative must satisfy identity (8). However, surprisingly, the converse is very nearly true. For the moment let us consider *groupoids* (i.e. nonempty sets with a single binary operation). Let us say a groupoid is *k-nice* if the product of any k elements is independent of the way the elements are ordered or associated. For example, 2-nice is equivalent to being commutative. Being both 2-nice and 3-nice is equivalent to being commutative and associative. Experiments with **Albert** suggested that in the presence of identity (8), an algebra's multiplication is 5-nice. This provided the needed courage to prove in [7] that

Theorem 3 *A groupoid satisfying $(xy)z = y(zx)$ is k-nice for each $k \geq 5$.*

Recall that an algebra is *semiprime* if $I^2 \neq 0$ for any nonzero ideal I. Theorem 3 can then be used to prove the following theorem, which generalizes a result of Chen.

Theorem 4 *A semiprime algebra satisfying $(xy)z = y(zx)$ is commutative and associative.*

4 Generalizing a result of Paul

In [18], Paul studied algebras satisfying

$$(a,b,c) - (a,c,b) = 0 \tag{9}$$
$$(a,[b,c],d) = 0 \tag{10}$$

Clearly these conditions form a generalization of associativity since, in an associative algebra, all of the associators are zero. An algebra is *prime* if $IJ \neq 0$ for any two nonzero ideals I and J. Paul showed that

Theorem 5 (Paul) *Any prime algebra satisfying both (9) and (10) is either associative or else its nucleus and center coincide.*

This theorem is part of a broad collection of related results providing sufficient conditions for an algebra to be commutative or associative (e.g. [22]). Without defining "nucleus" and "center", the reader sees that Paul's theorem is clouded by the two possibilities appearing in the conclusion, and is therefore somewhat unsatisfying.

Theorem 6, shown below, was proven with the help of **Albert**. It generalizes Paul's result in two ways. First, it removes the ambiguity by eliminating the second condition in the conclusion of Paul's theorem. And second, it assumes only that $\mathcal{A}$ is semiprime, an assumption weaker than that of being prime.

Theorem 6 *Any semiprime algebra, having characteristic* $\neq 2, 3$ *and satisfying (9) and (10), is associative.*

Albert's operators, and the ease with which they can be composed, are well-suited for studying laws such as Paul's identities. To enter identity (10), for example, the user need only enter:

```
identity (a,[b,c],d)
```

rather than enter its expanded form:

```
(a(bc))d - (a(cb))d - a((bc)d) + a((cb)d)
```

Theorem 6 was arrived at using a sequence of several lemmas. Initially the lemmas were established with **Albert**. Once **Albert** had confirmed their validity, Kleinfeld was able prove the lemmas in about five pages using traditional-style arguments. This experience suggests a top-down methodology whereby the researcher first sketches out a "high-level" proof using a sequence of unproven lemmas. Next, using **Albert**, the lemmas are confirmed. Finally, being confident of their truth, the researcher seeks conventional proofs. A description of this case study can be found in [6].

5 Verification of the Miheev question

Algebras satisfying

$$(x, y, y) = 0 \tag{11}$$

are called *right alternative*. The right alternative identity, of course, is another generalization of associativity. The well-known Cayley algebras (see [20]) satisfy this identity (along with its counterpart the left alternative law.) In [17], Miheev proved that any right alternative algebra also satisfies

$$(y, y, x)^4 = 0,$$

but, in general,

$$(y, y, x)^2 \neq 0.$$

The question remained whether $(y, y, x)^3 = 0$. In 1987 Hentzel and Jacobs used a combination of group representation theory and novel sparse matrix methods to show that in free right alternative algebras $(y, y, x)^3 \neq 0$ [10]. This project took several months of effort, and probably a month of computing time. **Albert**'s efficiency is evidenced by the fact that it can now affirm this result in about an hour on a typical university workstation!

6 Weaknesses and future work

We feel the results obtained so far justify the effort required in **Albert**'s development. Still, **Albert** has a number of shortcomings, and there is more work to do. Two major shortcomings of **Albert** have to do with characteristic zero and small prime characteristics. The first of these problems is, simply, that at present **Albert** has no support for rational arithmetic.

Next, **Albert** is not satisfactorily equipped to handle small prime characteristics. **Albert** requires that all defining identities be *homogeneous*. Moreover, defining identities are always treated in their linearized form. For example x^2 is treated as $xy + yx$. Over fields of sufficiently large characteristic, the two identities are equivalent. Thus, for characteristic greater than 2, $x^2 = 0$ is the same as $xy + yx = 0$. At present, our algorithm depends on a linearized defining identity. Hence, this problem requires further study.

More generally, the theory of nonassociative R-algebras can be developed over arbitrary commutative rings R with unity [23]. Can the methods employed by **Albert** be extended to this setting?

Another improvement would be in the case when f is found to be an identity, and allow the system to emit a real proof of this fact. **Albert** could be made more efficient by exploiting properties for certain common identities. For example, with commutativity there is a symmetry that could be exploited. Similarly, the associative law could be exploited to greatly simplify computations. Finally, another opportunity for improvement might be a graphical user-interface.

It is known that random substitution methods can be used to decide if a given finite dimensional algebra satisfies a certain identity [13]. But can random methods be used for the varietal problem described in this paper?

Albert is written in about 13,000 lines of C, and is free. It may be obtained by ftp'ing to `ftp.cs.clemson.edu`, typing `anonymous` at the login prompt, and then downloading the source modules in the albert directory.

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